or more accurately, the "eyes" on these slides, when they are in line with the hinge pins. Then the slides which are on the arms, and carry the drums D, D_1, D_2, will be to the right of S, P, Q, respectively. Let R be a weight at some fixed point on the cord which is attached to the drum D_2 and whose lower end is fastened to the main bar.

Move the slide at S down to S_1, where it will remain fixed, making

$$SS_1 = a. \tag{2}$$

This will let all the slides move downward the same distance, the slide at P stopping at P_1, making $PP_1 = a$, also.

Now turn the first arm through an angle θ, the slide on it taking the position D, with the cord S_1D perpendicular to the arm SD. Since the cord attached to the drum D passes through the eye on the slide at S_1, and is made fast to the slide at P_1, this will pull the latter up to P_2, making

$$P_1P_2 = S_1D = a \sin \theta.$$

$$\therefore \quad PP_2 = PP_1 - P_1P_2 = a(1 - \sin \theta).$$

Next unwind a length b on the cord attached to the drum D. This will let slide P_2 move down to P_3, and since the slide Q has moved in precisely the same way as P, taking the same number of corresponding positions, and is now at Q_3, we have

$$PP_3 = QQ_3 = a(1 - \sin \theta) + b. \tag{3}$$

Then turn the second arm through the same angle θ, the slide on it taking the position D_1. Since the cord attached to D_1 passes through the eye on P, and is fastened to Q, this will pull Q up to Q_4, making

$$Q_3Q_4 = P_3D_1 = PP_3 \sin \theta.$$

$$\therefore \quad QQ_4 = QQ_3 - Q_3Q_4 = PP_3(1 - \sin \theta).$$

$$= a(1 - \sin \theta)^2 + b(1 - \sin \theta).$$

Now unwind a length c from drum D_1. This will move Q down to Q_5, and since the point R has moved in the same manner as Q, and is now at R_5, we have

$$QQ_5 = RR_5 = a(1 - \sin \theta)^2 + b(1 - \sin \theta) + c. \tag{4}$$

Then turn the third arm through the angle θ. Since the cord attached to D_2 passes through the eye on Q, this will draw the point R up to R_6, making

$$R_5R_6 = Q_5D_2 = QQ_5 \sin \theta.$$

$$\therefore \quad RR_6 = RR_5 - R_5R_6 = QQ_5(1 - \sin \theta),$$

$$= a(1 - \sin \theta)^3 + b(1 - \sin \theta)^2 + c(1 - \sin \theta).$$

Finally unwind (or wind up) a distance d at drum D_2. This will move R to R_7, making

$$RR_7 = a(1 - \sin\theta)^3 + b(1 - \sin\theta)^2 + c(1 - \sin\theta) + d. \tag{5}$$

If the angle θ be taken so that the last cord shall now be taut, the point R_7 will coincide with the initial position of R. We shall then have $RR_7 = 0$, and hence it is evident from equation (5), that $(1 - \sin\theta)$ will then be a root of the equation

$$ax^3 + bx^2 + cx + d = 0. \tag{6}$$

The values of $(1 - \sin\theta)$ for every degree from 0° to 90° are the numbers written on the circular scale under the first arm from which the roots are read.

Furthermore, from equations (2), (3), and (4), we see that the values of SS_1, PP_3, and QQ_5 are the results obtained by synthetic substitution, when $(1 - \sin\theta)$ is substituted for x in equation (6). So that the instrument not only gives a root, r, say, but at the same time the readings on the main bar give the quotient obtained when the equation is divided by $x - r$.

This fact was not pointed out by Professor Peddie. His method of proof does not reveal it. Although I can see no reason why his instrument should not give these coefficients, I do not believe these numbers could be read off on his machine, because all of his scales are placed on the drums.

The quotient given in the solution of equation (1) above was

$$10x^2 + 27.5x + 19.1.$$

Limits of the Mechanism. Since $(1 - \sin\theta)$ is a number that always lies between 0 and 1, the instrument will find only a root that lies between 0 and 1.

Furthermore, the distances SS_1, PP_3, and QQ_5, must all be positive. Otherwise it would be necessary for the slide on at least one of the arms to cross over to the other side of the main bar, which is manifestly impossible. Hence the coefficients of the quotient must all be positive. This means that the equation to be solved must have no positive root except the one lying between 0 and 1. (This is another fact that was not stated specifically by Professor Peddie, although it must hold for his instrument also.) Hence, the equation to be solved must have the constant term negative, and all the other coefficients positive; and the sum of the positive coefficients must be greater than the constant term.

An equation of the first degree can be solved by using only the upper, or lower, arm. An equation of the second degree can be solved by using two arms, either the upper two, or the lower two. In order to solve an equation of the fourth degree, it would be necessary to add another arm at R in Fig. 3, and so on, for equations of higher degree.

This mechanism can also be used to graph a function of x from $x = 0$ to $x = 1$. To do this it would only be necessary to attach the last cord to a slide at R, say, (Fig. 3) instead of making it fast to the main bar. Then "put the function on the instrument" just as in solving an equation. When the instrument is closed, the distance of this slide R from its initial position will be the

value of the function when $x = 1$, that is, the sum of the coefficients of the function. As the arms are turned from 0° to 90°, keeping the slides properly adjusted and all the cords taut, the distance of this slide, R, from its initial position will be continuously the value of the function as x varies continuously from 1 to 0.

ON THE ORTHOCENTRIC QUADRILATERAL.[1]

By NATHAN ALTSHILLER-COURT, University of Oklahoma.

Introduction. (*a*) The altitudes AD, BE, CF of a triangle ABC meets in a point H, the orthocenter of ABC. The triangle DEF formed by the feet D, E, F, of the altitudes is frequently called the orthic triangle of ABC.

To Carnot[2] is due the credit for having called attention to the almost obvious fact that *each of the four points H, A, B, C, is the orthocenter of the triangle formed by the other three.*

The points A, B, C, H are referred to as an *orthocentric group of points*, or an *orthocentric quadrilateral*, and the four triangles determined by these four points as an *orthocentric group of triangles*.

(*b*) In 1821 Brianchon and Poncelet showed that the circumcircle (N) of the orthic triangle DEF of ABC passes through the mid-points A', B', C', of the sides BC, CA, AB, of ABC, and also through the mid-points P, Q, R, of the segments AH, BH, CH respectively.[3] That the circle through the first six points mentioned passes also through the last three becomes obvious if we observe that *DEF is the orthic triangle not only of ABC, but of each of the four triangles of the orthocentric group $ABCH$.*

(*c*) In 1822 Feuerbach proved[4] that the circle (N) is tangent to the four circles which touch the sides of the triangle ABC. It was not until 1861 that Sir William R. Hamilton pointed out that (N) is also tangent to the circles touching the sides of the triangles BCH, CHA, HAB.[5] Now since the orthic triangle DEF is common to the four triangles of the orthocentric group $ABCH$, the circumcircle (N) of DEF is the nine-point circle of each of these four triangles, and therefore Hamilton's extension of Feuerbach's theorem becomes self-evident.

---

[1] Read before the American Mathematical Society, St. Louis, December 31, 1919. Readers of this article will be interested in comparing it with the first part of the author's earlier article "On the I-centre of a triangle" (*1918*, 241–246)—EDITOR.

[2] Carnot, *De la corrélation des figures de Géométrie*, 1801, p. 102.

[3] For the proof, see, for instance, J. Casey, *A Sequel to Euclid*, second edition, 1882, p. 58, or C. V. Durell, *Plane Geometry for Advanced Students*, vol. 1, pp. 30–31.

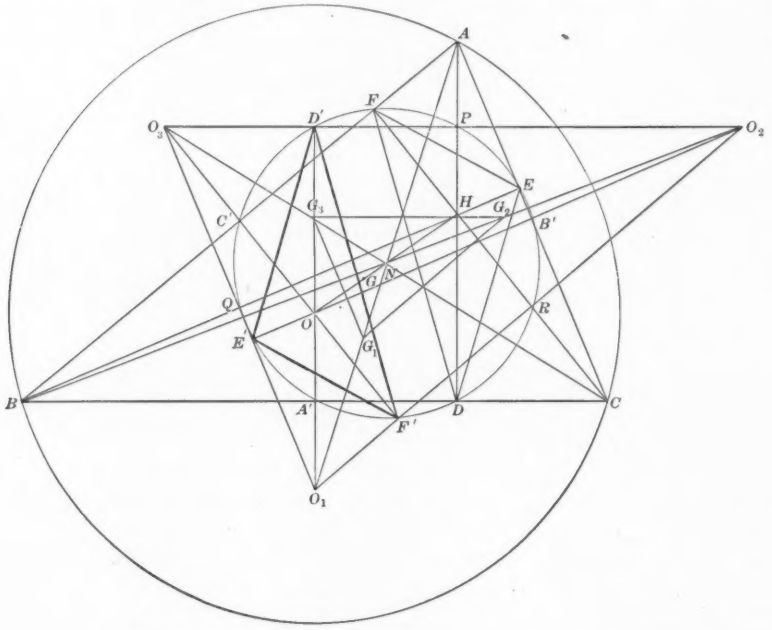
[4] For a proof see Casey, *l.c.*, pp. 58–61, or Durell, *l.c.*, pp. 46–47 and pp. 149–150.

[5] In making this statement Professor Altshiller-Court was possibly misled by Casey's reference to the result as "Sir William Hamilton's Theorem" (*Quarterly Journal of Mathematics*, 1861, p. 249) and by the fact that Sir William proposed the result as a problem in *Nouvelles annales de mathématiques*, 1861, vol. 20, p. 216.

The result was not, however, given originally by Sir William, but by T. T. Wilkinson, as prize-problem 1883 in *Lady's and Gentleman's Diary*, London, 1854, p. 72 (Solutions, *Diary*, 1855, pp. 67–69).—EDITOR.

These examples suggest that in certain connections it may be fruitful to consider the circle (N) as belonging not to the triangle ABC, but to the orthocentric group $ABCH$. The following considerations are based on this remark.

1. The center N of the nine-point circle (N) of the triangle ABC was shown by Benjamin Bevan, in 1804,[1] to lie midway between the orthocenter H and the circumcenter O of the triangle ABC. In other words, the circumcenter O of ABC is the symmetric of H with respect to N. But, as has been pointed out above, (N) is also the nine-point circle of the triangle BCH, whose orthocenter is the point A, hence the circumcenter O_1 of BCH is the symmetric of A with



respect to N. Similarly for the circumcenters O_2, O_3, of the triangles CHA, HAB. Consequently: *The four circumcenters of an orthocentric group of triangles form an orthocentric group which is the symmetric of the given group with respect to the nine-point center.*

2. From the symmetry of the two groups of points $ABCH$ and $O_1O_2O_3O$ follow immediately all the known properties of the circumcenters O_1, O_2, O_3, O. For instance:

(*a*) *The triangles $O_1O_2O_3$ and ABC are congruent*[2] and furthermore, *their sides are respectively parallel.* It may also be observed that these properties hold for the pairs of triangles O_2O_3O and BCH; O_3OO_1 and CHA; OO_1O_2 and HAB.

[1] For the proof compare Casey, or Durell, *l.c.*

[2] Durell, *l.c.*, p. 36, exercise 89.

(*b*) *The point H is the circumcenter of the triangle* $O_1O_2O_3$.[1] Similarly the points A, B, C, are the respective circumcenters of the triangles O_2O_3O, O_3OO_1, OO_1O_2.

(*c*) *The lines* BO_3 *and* CO_2 *are parallel*[1] and similarly for other pairs of analogous lines.

3. A wealth of other propositions, so far unobserved or unannounced, may be derived from the two symmetrical figures. We shall call attention to the following. In the symmetrical transformation considered the center of symmetry N is a double point, and the nine-point circle (N) is transformed into itself, hence: *An orthocentric group of triangles and the orthocentric group of their circumcenters have the same nine-point circle.*

4. The orthic triangle $D'E'F'$ of the orthocentric group $OO_1O_2O_3$ is the symmetric of the orthic triangle DEF of the orthocentric group $ABCH$, the pairs of points D, D'; E, E'; F, F'; being diametrically opposite on the circle (N). Thus we find a geometric interpretation of *three new points of the nine-point circle of the triangle ABC.*

5. The nine-point circle (N) of the orthocentric group $OO_1O_2O_3$ is tangent to the sixteen circles which touch the sides of the four triangles of this group, according to Hamilton's extension of Feuerbach's theorem (Introduction). These sixteen circles are the symmetric, with respect to N, of the analogous sixteen circles of the orthocentric group $ABCH$. Thus we find *sixteen new circles which are tangent to the nine-point circle of the triangle ABC.*[2]

6. The orthocentric group $OO_1O_2O_3$ has been derived by symmetry from the given orthocentric group $ABCH$. But the process may be reversed, and the group $HABC$ may be derived from the group $OO_1O_2O_3$ considered as given. Consequently: *The vertices of a given orthocentric group of triangles may be considered as the four circumcenters of a second orthocentric group of triangles, the two orthocentric groups having the same nine-point circle and being symmetrical with respect to its center.*

7. The point of intersection G of the medians of ABC, often referred to as the centroid of ABC, lies, according to a theorem of Euler,[3] on the line joining the orthocenter H to the circumcenter O of ABC, and we have $GO/GH = 1/2$. Since the nine-point center N is the midpoint of OH, we have $NG/NH = 1/3$, the points G and H being on opposite sides of N. In other words, the point G

[1] Durell, *l.c.*, p. 36, exercise 88.

[2] The results of paragraphs 1, 2, 3, 4 and 5 were given by T. T. Wilkinson in *Mathematical Questions with their Solutions from the Educational Times*, London, Vol. 1, 1864, pp. 6–7; see also *Mathematical Questions*, etc., Vol. 6, 1866, pp. 25–26.

T. T. Wilkinson seems to have been the first to discover an infinite series of circles tangent to the nine-point circle of a triangle (*Lady's and Gentleman's Diary*, London, 1857, p. 86; 1858, 87): "*If the radical centers of the inscribed and escribed circles of any triangle be taken, and circles be inscribed and escribed to the triangles formed by joining these radical centers, and the radical centers of the latter system of circles be again taken and circles inscribed and escribed to the triangles thus formed, and so on* ad infinitum, *the infinite number of circles thus found, as well as the original system of inscribed and escribed circles, always touch the circle drawn through the middle points of the* [sides of the] *first triangle.*"—EDITOR.

[3] Durell, *l.c.*, p. 41.

corresponds to H in a similitude of ratio $-1/3$, the center of similitude being N.[1] But N is also the nine-point center of the triangle BCH, whose orthocenter is A, hence the centroid G_1 of BCH corresponds to A in a similitude of ratio $-1/3$ with N as center of similitude. Similarly for the centroids G_2, G_3, of the triangles CHA, HAB. Consequently: *The four centroids of an orthocentric group of triangles form an orthocentric group, the two groups being similar and similarly placed.*

8. Since the centroids G, G_1, G_2, G_3, form an orthocentric group, all the properties of such a group immediately follow, as, for instance, that G is the orthocenter of the triangle $G_1G_2G_3$, etc.

Again the similitude of the two groups $GG_1G_2G_3$ and $HABC$ puts into evidence a great many properties, as for instance, that G_1G_2 is parallel to AB and is equal to 1/3 of its length; that the point of intersection of GG_1 and G_2G_3, which will be represented by (GG_1, G_2G_3), is collinear with N and $D \equiv (HA, BC)$; etc. The reader may find it interesting to formulate a number of these propositions.

9. In the similitude (7) by which the group $GG_1G_2G_3$ is derived from the group $HABC$, the center of similitude N is a double point. Hence: *An orthocentric group of triangles and the orthocentric group of their centroids have the same nine-point center.*

10. The orthocentric group $GG_1G_2G_3$ has been derived from the given orthocentric group $HABC$ by a similitude of center N and ratio $-1/3$. But the process may be reversed, and the orthocentric group $HABC$ may be derived from the orthocentric group $GG_1G_2G_3$, considered as given, by a similitude of ratio -3, the center remaining the same. Consequently: *The four points of an orthocentric group may be considered as the centroids of another orthocentric group of triangles, the two groups having the same nine-point center, this point being a center of similitude of the two groups, the ratio of similitude being* -3.

11. Since from (1) the two groups $HABC$ and $OO_1O_2O_3$ are symmetrical about the center N, therefore it follows from (10) that the two groups $GG_1G_2G_3$ and $OO_1O_2O_3$ admit N as a center of similitude, the ratio of similitude being $+3$. Hence: *The centroids and the circumcenters of an orthocentric group of triangles form two orthocentric groups of points having the same nine-point center, this point being a center of similitude of these two groups, the ratio of similitude being* $+3$.

1720

C. Maclaurin's *Geometria organica sive descriptio linearum curvarum universalis*, published at London—G. Poleni's *De mathesis in rebus physicis utilitate praelectio habita* $\cdots$, published at Patavia—Second edition of L'Hospital's *Traité analytiques des sections coniques*, published at Paris—Alexandre Savérien, author of *Dictionnaire universel de mathématiques et de physique* (2 vols., Paris, 1753), born July 16.

[1] Euler, *Novi comment. acad. sc. Petrop.*, vol. 11 (1765), 1767, p. 114.—Editor.

A GRAPHIC SOLUTION OF THE CUBIC EQUATION.[1]

By J. P. BALLANTINE, University of Maine.

The purpose of this paper is to suggest a graphic method of finding the real roots of a numerical cubic equation, a method which will involve comparatively little work when there are a large number of problems.

Let us consider the general cubic reduced to the standard form

$$u^3 + u^2 + xu = y. \tag{1}$$

This can easily be effected by dividing by the coefficient of u^3, and then by dividing the roots of the equation by the coefficient of u^2. It may happen in a particular case that this latter coefficient is zero, in which case the form obtained will differ from the form given above in that the u^2 term is missing; but a similar discussion is easily worked out for that case.

All the cubics of the standard form are easily set in a one to one correspondence with the points of the plane. Let us say that the x and the y of the cubic are the coördinates of the corresponding point.

The necessary and sufficient condition that v be a root of the standard cubic is that:

$$v^3 + v^2 + xv = y. \tag{2}$$

The same is a necessary and sufficient condition that the point (x, y) lie on a certain straight line, of which the above is said to be the equation. The points therefore corresponding to a set of isoradical cubics[2] lie on a straight line, and the common root of the cubics is the slope of the line.

Let us next consider the locus of points corresponding to cubics with two equal roots. The necessary and sufficient condition that v be a double root of

[1] While the essential features of this solution are not new they are probably sufficiently unfamiliar to readers of the MONTHLY to render Mr. Ballantine's presentation of his rediscovery of them a matter of general interest. As applied to the equation $u^3 + xu + y = 0$ there is a recent discussion in J. Lipka, *Graphical and Mechanical Computation*, 1918, pp. 35–36. A similar discussion is found in C. Runge, *Graphical Methods*, New York, 1912, pp. 59–60 (also *Praxis der Gleichungen*, Leipzig, 1900, p. 159), and L. E. Dickson incorrectly attributes the method to Runge (Dickson, *Elementary Theory of Equations*, New York, 1914, p. 17). This method originated with Léon Lalanne who made the discovery in 1843 and regarded the cubic as a special case of the trinomial equation $u^m + xu^n + y = 0$ ("Sur les tables graphiques et sur la géométrie anamorphique . . .," *Annales des ponts et chaussées*, 1846, 1er semestre, pp. 27–29; *Comptes rendus de l'académie d. Sc.*, Paris, tome 81, 1875, pp. 1187–1188; and "Méthodes graphiques pour l'expression des lois à trois variables," *Notices réunies par le ministere des travaux publics à l'occasion de l'Exposition universelle de Paris*, 1878). Lalanne found that his form of the cubic equation led to the semi-cubical parabola $4x^3 + 27y^2 = 0$ as an envelope.

The subject is also discussed in J. de la Gournerie, *Traité de Géométrie descriptive*, seconde édition, troisième partie, Texte, Paris, 1885, pp. 211–212, Atlas, 3, fig. 455 (filling a quarto page), plate 46; and in M. d'Ocagne, *Traité de Nomographie*, Paris, 1899, pp. 43–44.—EDITOR.

[2] Two cubics are said to be isoradical if they have a root in common. A straight line is said to be isoradical if its points correspond to isoradical cubics.

the standard cubic is that:

$$(3) \qquad \begin{cases} v^3 + v^2 + xv = y \\ 3v^2 + 2v + x = 0. \end{cases}$$

This is also the necessary and sufficient condition that the point (x, y) lie on the envelope of the straight line. Every isoradical straight line is therefore tangent to the locus

$$(4) \qquad 4x^3 - x^2 + 18xy + 27y^2 - 4y = 0$$

obtained by eliminating v from (3).

All that is necessary, therefore, in order to solve a cubic, is to reduce it to standard form, obtaining the quantities x and y. Plot the corresponding point, and drop the three tangents to the locus (4). The slopes of the three tangents will be the three roots of the equation as written in standard form.

Note that the locus (4) can be plotted once for all, and in place of plotting a cubic equation, one need only plot a point for each problem to be solved.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

Psychologists today are restoring to honor, albeit in much modified form, the Old Reliable Dream Book. Theologists and educators are seeking to read the future. It is not surprising that a claim should be made for the use of mathematics as a means of prediction. Such a claim is stated by Professor Weaver in the first discussion this month. His conception must not be dismissed as fantastic without an investigation into its meaning. He makes quite clear that no prediction of individual events or circumstances is intended; his hope is at most that the progress of history in the large or average sense may be forecasted. In support of his contention may be cited the service of mathematics in exactly this function of prediction in connection with the physical sciences. It may even be argued that an example of similar application to the social sciences is to be found in the generally accepted theory of the periodicity of economic cycles of prosperity and depression, with an approximate period of ten or eleven years. Against this argument may be placed the apparently hopeless complexity, as regards mathematical formulation, of the social problem, compared with the physical problem. It is doubtful whether the reduction to concise qualitative laws, or even the specification of the independent variables in terms of which such laws are to be stated lies within the mental potentialities of the human race.

Several details in Professor Weaver's account suggest comment. For example, it is not obvious that the analogy of the determination of all values of an everywhere analytic function by the values of the function and its derivatives

at a single point, is the most natural one. Would it not be more appropriate, in view of the suggestions of mathematical physics, to suppose that the history of the universe is contained in a vast system of differential equations, or in view of Volterra's hypothesis of "heredity," of integro-differential or other still more complicated equations, so that the knowledge of a *finite* set of initial conditions would suffice to determine a solution completely?

Also the notion of the "most probable" of all possible trends of natural phenomena may be much less simple than special cases would indicate. That determination of the history of a country or the world which will produce the "most probable" of all conceivable birth-rates may not coincide with the determination which will produce the "most probable" duration of life. This almost obvious comment is not meant as an objection to Professor Weaver's remarks, nor is it implied that a "most probable" configuration under the whole imaginable set of auxi iary conditions may not also exist; such questions would naturally have to be settled if the present vision should come within appreciable range of reality.

The second discussion brings us back to something more concrete. Dr. Sensenig indicates a form of derivation of the integral as the limit of a sum, which practically amounts to a proof, for the case of an analytic function, of the identity of the concepts *integral* and *anti-derivative*. It is at times useful to have such proofs at hand, even in elementary instruction, as aids in producing conviction in the minds of students. Of course, the use of the formula for Σn^m will prove an obstacle to the use of Dr. Sensenig's work for that purpose. It is doubtless universally familiar that the convergence of the Cauchy sum, in the case of a monotonically increasing or decreasing continuous function, can be cast in a geometric form so vivid as to be accepted with ease by the ordinary classes in calculus.

As the third discussion appears an alternative derivation of the expressions for the half-angles of a plane triangle in terms of the sides. Professor Baudin obtains the relations from the law of sines instead of from the law of cosines. The work is less simple than the ordinary methods; it is interesting, however, to see that it can be carried out.

I. Forecast.

By Warren Weaver, California Institute of Technology.

The foundations of science and scientific thought have been subjected in the last fifty years to examination of a most critical sort. This examination has resulted in development along two general lines; a filling of needful matter into the interstices of the even yet porous body of logical thought where the general trend of the previously accepted body has been found still tenable, and an opening up of new problems in those regions where the old theories have been found inadequate or merely approximate. As an example of the first type there comes to one's mind the rigor that has been brought to the fundamental concepts of the calculus by means of function theory. The outstanding illustration of the latter

type of problem of the newer science is the recasting of the old concepts of time and space that is necessary under the modern theory of relativity, along with the fundamental and far-reaching alterations which this theory imposes upon mechanics and electrical theory.

In many such investigations a mode of attack is being used that reminds one of the older metaphysical and theological considerations of scientific problems, such as the proof of the Principle of Least Action given by Maupertius. As the type of problem being studied becomes more and more fundamental and ultimate in nature one is not surprised to find that philosophical considerations enter along with considerations of a more formal mathematical nature. So that it is a little less idle than hitherto for one to speculate how fundamental the truths really may be that he meets with in pure mathematics and in the mathematics of pure physics. Some of the theories that have hitherto seemed restricted to the formal considerations of the mathematician have indeed been emerging of late as truths the generality and broad applicability of which are not yet fully appreciated. Out of an almost indecent past shadily connected with the gaming table comes a theory of probability that, expanded by the researches of Gibbs and others, furnishes the one ultimate foundation for the kinetic theory of gases, kinetic theories of electrical conductivity, a broad class of thermodynamical problems, and the theory of statistics in all its branches. The fact that of all possible complexions of a system the trend of nature is always towards the most probable seems to be the statement of a principle whose breadth has been only of late and perhaps not even now fully appreciated. One may easily cite other modern questions to which one can give a broadly philosophical cast. As has been suggested in a recent note in *Science*[1] velocity is essentially of such nature that there is a *best* way to measure it, a conclusion which, if completely acceptable, may be taken as a fundamental argument for the relativity theory. The work of Dr. R. C. Tolman in developing what he has called the Principle of Similitude[2] has furnished upon some such general ground not only a startlingly immediate method of establishing many physical relationships known upon other grounds to be correct, but seems also to disclose these results as *inevitable* truths in a way that a more formal proof does not at all. The atomic structure of matter and the corpuscular nature of energy seem to be demanded by the fact that if nature is to tend towards some certain definite most probable configuration and a certain definite most probable distribution of energy it must be possible to recognize discrete elements of matter and energy.[3] Generalized relativity has been extended along certain lines to suggest that it may be the expression of a very fundamental truth that possibly and probably transcends the sciences of mathematics and physics, narrowly conceived, and is applicable to diversified considera-

[1] "The Nature of Velocity" by Tenney L. Davis, *Science*, Oct. 10, 1919.

[2] Articles on the Principle of Similitude and related subjects: *Physical Review*, April, 1914; Aug., 1914; *Phys. Rev.*, 6, 219 (1915); July, 1916; *Phys. Rev.*, 9, 237 (1917).

[3] *Eight Lectures on Theoretical Physics* by Max Planck, pp. 42–55. Columbia University Press.

tions regarding life.[1] Dr. H. Bateman has suggested, in the note in the *Philosophical Magazine* to which reference has just been made, the possible and probable existence of a deep underlying principle which, since it will be the ultimately true equation of motion, not only would furnish the results of ordinary relativity theory, but would also contain, among other things, an explanation of the fact that good designs are perpetuated in nature. We may conclude therefore that the trend of scientific thought at the present time renders it more hopeful and less inexcusable to contemplate analogies between the principles of mathematics and the principles of life, in the attempt to enrich and perhaps shed new light upon each.

It ought to seem queer to the scientist that history is so nearly exclusively a record of what man has done, and in so very small measure a study of what man will do. For the method of the scientist is to record past events only so far as is necessary to the accumulation of sufficient data to check theories which make possible the prediction of events in the future. It has probably been many times suggested that serious study be made of the possibility of a similar method being used in history, but perhaps the best-known comes from the versatile pen of H. G. Wells. In a discourse entitled "The Discovery of the Future"[2] delivered at the Royal Institution in London he gives his foundation for an optimistic hope that a bold and searching study may serve to give us eventually a good deal of general but exceedingly useful knowledge of the future. He says that "as one assimilates the broad conception of science, the persuasion comes into one's mind that the adequacy of causation is universal," so that "the man of science comes to believe at last that the events of the year A.D. 4000 are as fixed, settled, and unchangeable as the events of the year A.D. 1600." He cites the pushing back of our information to prehistoric times by means not of revelation, but by means of a keen and vigorous habit of inquiry, and continues; "and now if it has been possible for men by picking out a number of suggestive and significant looking things in the present, by comparing them, criticizing them, and discussing them, with a perpetual insistence upon 'Why?', without any guiding tradition and indeed in the teeth of established beliefs, to construct this amazing searchlight of inference into the remoter past, is it really, after all, such an extravagant thing to suggest that, by seeking operating causes instead of fossils and by criticizing them as consistently and thoroughly as the geologic record has been criticized it may be possible to throw a searchlight of inference forward instead of backward, and to attain a knowledge of coming things as clear, as universally convincing, and infinitely more important to mankind than the clear vision of the past geology has opened to us during the nineteenth century?" The lecture includes a discussion to show that the existence of the "exceptional man," such as Napoleon, Cæsar, or William the Conqueror need not hopelessly remove the possibility for success of such a scheme, and it is careful not to hold out any

[1] See, for example, the end of an article called "The Physical Aspect of Time," by H. Bateman, *Memoirs and Proc. of Manchester Literary and Philosophical Soc.*, Vol. 54, Part 3, 1909–1910: and, same author, *Phil. Mag.*, Vol. 37, Feb., 1919.

[2] Published by B. W. Huebsch, New York, in book form with the above title.

fantastic hope for individual prophecy—a fact which need no more disturb the main result than the fact that we are unable to predict the motion of any single molecule in a gas need preclude the possibility of a statistical forecast of a future state of it. We can no more hope to forecast the life of a single certain man than, to extend Wells's illustration, we can hope to give a detailed history of the past life of some single certain labyrinthodon or pterodactyl.

A certain crude sort of mathematical justification for some such notion had occurred to the writer about a year before he chanced to see this book of Wells. An analytic function y of an independent variable x may be expanded by means of Taylor's Theorem in a power series in $(x - a)$. If this series represents the function y for values x to the left of a, it will also represent y for values of x to the right of a. In fact, the series will represent the function y for *all* values of x if it is convergent for all real values of x. And all we have to know in order to possess complete information about this series is the value of the function and of its derivatives *at the point a*. If we call the independent variable t instead of x the analogy becomes apparent. It clearly suggests that if we could know, about a certain social factor, its present state, and have complete instantaneous information as to the trend of it we ought to be able to forecast accurately its importance at any future time.

One may conjecture some general things about certain types of functions that would probably prove necessary in such a connection. A general steady growth would be represented by a linear function, to determine which we would need only its present value, and present rate of change. The sort of thing of which one says that "history repeats itself" would be represented by circular or other periodic functions. One may guess that several such, of different periods, would combine to produce nearly zero result for long stretches of time, and then with a sudden "beat," through constructive addition, give a sharp maximum—perhaps we may say a Renaissance! Any factor which is being damped out would in all probability be represented by an ordinary exponential damping term, with negative exponent. It is conceivable that little else would be necessary, and in all these cases it is cheerful to note that our information would extend to *any* time in the future, however remote; which corresponds directly to the mathematical fact that the Maclaurin expansions of the functions kt, $\sin t$, $\cos t$, and e^{-kt} are convergent for all real values of t. We might conceive of a Fourier series (perhaps multiplied by an exponential factor) doing the trick for us, so that some later-day genius could construct for us a tide-predicting machine not for the ocean's tides, but for the tides of human thought and the surge of human progress.

A possible objection to the application of the above analogy lies in the fact that one needs, in general, the value of an infinite number of derivatives at the point a really to possess the whole series. Beyond the fact that a finite number of terms (and often a small number) gives a good approximation for short ranges of x mathematics has a further hopeful suggestion. In the potential theory it is very well known that if we have given, over a closed surface S, the value at every point of the potential function V we can calculate the potential at any other

point.[1] In terms of our former analogy we may say that if we have a set of "cross-section" values of a function (which in our application would be a set of values for a constant t) and of its first derivative, and know that the function is characterized and specialized by the fact that it satisfies some very general condition (which condition while very general must get at the very root of the matter) we can completely determine the function at all other points. This general condition, in the case of ordinary potential and many allied problems is, of course, given by the ordinary Laplacian equation $\Delta^2 V = 0$. This suggests that if, by careful study, we could but obtain some universally applicable and fundamentally far-reaching principle—some truly dynamic theory of history, to use Henry Adams's phrase, we could probably thus cut down enormously upon the amount of contemporary data necessary.

This analogy may perhaps all be interpreted as merely strengthening Wells's point that in science effect seems inevitably to follow cause in an unending and inescapable chain of events. There is perhaps no more striking example of the adequacy of causation than that contained in the fact that the value and behavior of y a million million miles down the x axis is inevitably determined by the way in which the curve crosses the y axis; and it is a natural and, it would seem, not ridiculous hope to expect that some day there will be an historian bold enough and wise enough to write down a few of the equations of history which will hold not only for the minus sign, but also for the plus.

II. A Proof of the Definite Integral Formula.

By Wayne Sensenig, Conshohocken, Pa.

On pages 214–216 of Lamb's *Infinitesimal Calculus* several special cases of the definite integral are calculated "ab initio."[2] The following direct proof of the formula for the definite integral covers cases where y can be expressed as a *convergent* power series in x over the *closed* interval considered in the definite integral. The area under a curve is used as an illustration, but the form of proof is obviously independent of any special geometric interpretation. Use is made of the following well-known formula:[3]

$$\begin{aligned} S_m &= 1^m + 2^m + \cdots + n^m \\ &= \frac{n^{m+1}}{m+1} + \tfrac{1}{2}n^m + B_1 \frac{m}{2!} n^{m-1} - B_3 \frac{m(m-1)(m-2)}{4!} n^{m-3} \\ &\quad + B_5 \frac{m(m-1)(m-2)(m-3)(m-4)}{6!} n^{m-5} - \cdots, \end{aligned}$$

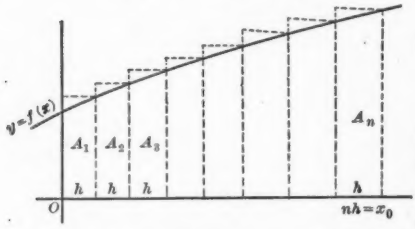
[1] If it is desired to calculate at points exterior to S the regularity of V at infinity is also necessary. The above mathematical fact does not correspond perfectly to the previous example, but the aspect of the logical content with which we are concerned is closely enough connected with the former to serve our purpose.

[2] Similar exercises are found in various text-books. Problems of the same character have from time to time appeared in the Monthly.—Editor.

[3] Hall and Knight, *Higher Algebra*, p. 337.

ending with a term in n or n^2; the B's being the Bernoulli numbers:

$$B_1 = \tfrac{1}{6}, \quad B_3 = \tfrac{1}{30}, \quad B_5 = \tfrac{1}{42}, \quad B_7 = \tfrac{1}{30}, \quad B_9 = \tfrac{5}{66}, \quad \cdots.$$



Let the interval of integration be $0 \leqq x \leqq x_0$, and suppose

$$y = f(x) = a_0 + a_1x + a_2x^2 + \cdots,$$

the series converging in an interval extending beyond x_0. Then, since the convergence must be uniform in the closed interval $|x| \leqq x_0$, any given ϵ, however small, will determine a value of m, such that

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m + R_m(x),$$

where

$$|R_m(x)| < \frac{\epsilon}{2x_0}, \qquad 0 \leqq x \leqq x_0.$$

Divide the interval into n equal parts each of length h, and form the Cauchy sum in the usual way. With reference to the accompanying figure, we have:

$$A_1 = a_0h + a_1h^2 + a_2h^3 + \cdots + a_mh^{m+1} + hR_m(h),$$

$$A_2 = a_0h + 2a_1h^2 + 2^2a_2h^3 + \cdots + 2^ma_mh^{m+1} + hR_m(2h),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_n = a_0h + na_1h^2 + n^2a_2h^3 + \cdots + n^ma_mh^{m+1} + hR_m(nh).$$

Thus

$$A = A_1 + A_2 + \cdots + A_n = s_{m,\,n} + hr_{m,\,n},$$

where

$$s_{m,\,n} = a_0nh + a_1h^2S_1 + a_2h^3S_2 + \cdots + a_mh^{m+1}S_m,$$

$$r_{m,\,n} = h[R_m(h) + R_m(2h) + \cdots + R_m(nh)].$$

Evidently

$$|r_{m,\,n}| < hn\frac{\epsilon}{2x_0} = \frac{\epsilon}{2};$$

so that independently of n, $r_{m,\,n}$ can be made arbitrarily small.

Furthermore, we have

$$nh = x_0$$

$$\lim_{n\to\infty} h^2S_1 = \lim_{n\to\infty} h^2\left(\frac{n^2}{2} + \frac{n}{2}\right) = \frac{x_0^2}{2},$$

$$\lim_{n\to\infty} h^3S_2 = \lim_{n\to\infty} h^3\left(\frac{n^3}{3} + \frac{n^2}{2} + B_1\frac{2}{2!}n\right) = \frac{x_0^3}{3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lim_{n\to\infty} h^{m+1}S_m = \lim_{h\to\infty} h^{m+1}\left(\frac{n^{m+1}}{m+1} + \frac{n^m}{2} + \cdots\right) = \frac{x_0^{m+1}}{m+1};$$

and therefore

$$\lim_{n\to\infty} s_{m,\,n} = a_0x_0 + a_1\frac{x_0^2}{2} + \cdots + a_m\frac{x_0^{m+1}}{m+1}.$$

The last expression is the sum of $(m+1)$ terms of the series

$$a_0x_0 + a_1\frac{x_0^2}{2} + a_2\frac{x_0^3}{3} + \cdots.$$

If we choose n so great that $s_{m,\,n}$ differs from its limit by less than $\epsilon/2$, then since $|r_{m,\,n}| < \epsilon/2$ for all n, we see that A differs from the sum of $(m+1)$ terms of the above series by less than ϵ. Since the series is readily shown to converge, it follows that the limit of A, as n becomes infinite, exists and is equal to the value of the series:

$$\int_0^{x_0} f(x)dx = a_0x_0 + a_1\frac{x_0^2}{2} + a_2\frac{x_0^3}{3} + \cdots.$$

The result can easily be extended to the case in which the lower limit of integration is not zero.

III. On a Formula of Plane Trigonometry.

By M. C. Baudin, Miami University.

The formulas for finding the angles of a triangle when the three sides are given, namely:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}},$$

where s is one half the perimeter of the triangle, may be derived trigonometrically from the law of sines in the following manner.

Consider the equations:

$$(1) \qquad \frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \frac{s-c}{s},$$

$$(2) \qquad \frac{\sin A - \sin B + \sin C}{\sin A + \sin B + \sin C} = \frac{s-b}{s},$$

$$(3) \qquad \frac{-\sin A + \sin B + \sin C}{\sin A + \sin B + \sin C} = \frac{s-a}{s}.$$

Let us change the left-hand members of these equations into quotients of products. The numerator of (1) may be written

$$(4) \qquad \sin A + \sin B - \sin C + \sin(A+B+C).$$

Now

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2},$$

$$\sin (A+B+C) - \sin C = 2 \sin \frac{A+B}{2} \cos \left(C + \frac{A+B}{2} \right)$$

and (4) becomes

$$2 \sin \frac{A+B}{2} \left[\cos \frac{A-B}{2} + \cos \left(C + \frac{A+B}{2} \right) \right]$$

or

$$4 \sin \frac{A+B}{2} \cos \frac{B+C}{2} \cos \frac{A+C}{2},$$

or

$$4 \cos \frac{C}{2} \sin \frac{A}{2} \sin \frac{B}{2}.$$

The denominator of (1) may be written

$$\sin A + \sin B + \sin C - \sin (A+B+C).$$

Changing $\sin A + \sin B$ and then $\sin C - \sin (A+B+C)$ into products and adding the results, we get

$$2 \sin \frac{A+B}{2} \left[\cos \frac{A-B}{2} - \cos \left(C + \frac{A+B}{2} \right) \right]$$

or

$$4 \sin \frac{A+B}{2} \sin \frac{B+C}{2} \sin \frac{A+C}{2}$$

or

$$4 \cos \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2}.$$

We therefore have

$$\frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \tan \frac{A}{2} \tan \frac{B}{2}.$$

In the same way we obtain

$$\frac{\sin A - \sin B + \sin C}{\sin A + \sin B + \sin C} = \tan \frac{A}{2} \tan \frac{C}{2},$$

$$\frac{-\sin A + \sin B + \sin C}{\sin A + \sin B + \sin C} = \tan \frac{B}{2} \tan \frac{C}{2}.$$

Substituting these values for the left-hand members in (1), (2) and (3), and solving those equations, we get

$$\tan \frac{A}{2} = \pm \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \tan \frac{B}{2} = \pm \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan \frac{C}{2} = \pm \sqrt{\frac{(s-a)(s-b)}{s(s-c)}},$$

The positive sign must evidently be chosen before the radical, since $A/2$, $B/2$, $C/2$ cannot be greater than $\pi/2$.

RECENT PUBLICATIONS.

REVIEWS.

Mathematical Philosophy and Philosophy of Mathematics.

Introduction to Mathematical Philosophy. By Bertrand Russell. New York, The Macmillan Company, 1919. 8 + 208 pages. Price $3.00.

The title of this book does not sufficiently indicate, perhaps no brief title could indicate clearly, the nature of the work. It is not so much an introduction to mathematical philosophy as to the philosophy of mathematics. The two 'philosophies' are very different things, the former being identified mainly by its method, the latter by its matter. An example of mathematical philosophy is the famous attempt of Spinoza to geometrize the philosophy of Descartes or his yet more famous attempt to construct the *Ethics* after the manner of Euclid. On the other hand, the philosophy of mathematics, as Mr. Russell himself here and elsewhere avows, is primarily concerned with such unanswered questions as arise in reflecting upon the nature of mathematics and the character of its foundations. This distinction being understood, we may say that there are two respects in which the present book may be properly said to be an introduction to the philosophy of mathematics. These respects may be indicated briefly as follows. When a question belonging to the philosophy of mathematics is at length acceptably answered, it ceases to be a question; the matter passes from the domain of philosophy to that of science, it becomes a genuine part of mathematics itself. Students of the logical foundations of mathematics are well aware, though the average professional philosopher is slow to learn, that this transition from speculation to knowledge has been a common and striking phenomenon in recent times. Many questions regarding the nature of such fundamentals as class, relation, number, continuity, infinity, and kindred matters, have been answered satisfactorily. The answers together with the ways of discovering them constitute the most fundamental branch of mathematical science. In the book before us Mr. Russell has endeavored to give the reader a stimulating acquaintance with some of these scientific results and methods. In this respect, then, the book is an introduction to the philosophy of mathematics, for the best approach to questions that have not been answered lies through acquaintance with the answers to those that have been answered. But the author has done more than that. Not only is the reader led in various directions *towards* the frontier of knowledge in this field but in several directions he is led *to* the frontier and is made to feel the challenge of unanswered questions that arise there. What is the characteristic—the necessary and sufficient criterion—of a mathematical proposition? What is to be the ultimate form of the theory of "logical types"? Is the multiplicative axiom, which is equivalent to that of Zermelo, true? In other words, is it true that, given any class of mutually exclusive classes, of which

none is empty, there is at least one class having exactly one term in common with each of the given classes? Is every infinite class reflexive? Such are some of the frontier questions which Mr. Russell's introduction makes the reader aware of.

The scope and content of the book are fairly indicated by the titles of its 18 chapters, which are as follows:

(1) The series of natural numbers; (2) Definition of number; (3) Finitude and mathematical induction; (4) The definition of order; (5) Kinds of relations; (6) Similarity of relations; (7) Rational, real and complex numbers; (8) Infinite cardinal numbers; (9) Infinite series and ordinals; (10) Limits and continuity; (11) Limits and continuity of functions; (12) Selections and the multiplicative axiom; (13) The axiom of infinity and logical types; (14) Incompatibility and the theory of deduction; (15) Propositional functions; (16) Descriptions; (17) Classes; (18) Mathematics and logic.

In the space allotted to this review it is impossible to present a digest of these chapters, for each of these is itself only a semi-popular digest, in some cases only a very compact and somewhat meager digest, of the more elaborate and more systematic and technical handling of the same topics in Whitehead and Russell's *Principia Mathematica* and in Russell's earlier work, *The Principles of Mathematics*. Accordingly I shall not attempt to offer a digest of these various digests but shall content myself with giving a sketchy account of some of them in the hope that the reader may be thereby drawn to the book itself and by it to the mentioned *Principia* and the great spiritual enterprise it represents.

The supreme thesis of the book is that logic and mathematics, rightly understood, are not two distinct sciences but that they together literally constitute one science without any inner division or breach from the most primitive things in what is traditionally called logic to the latest developments in what is traditionally called mathematics. Why, then, does Mr. Russell open his discussion with a consideration of the natural numbers, or ordinary integers? Are these assumed to be the most primitive of known things in logic (*i.e.* mathematics)? Far from it. In the *Principia* one must march through more than 350 large pages of solid developments before the concept of even the first of the integers is encountered. In an introduction designed to attract instead of repelling the uninitiated it was necessary to take most of those developments for granted; it was necessary to begin with something familiar; and nothing is more familiar than the so-called natural numbers, notwithstanding the fact that very few people, even among mathematicians, have taken the pains to ascertain the significance of these numbers in terms of logical primitives.

Mr. Russell's discussion of the natural numbers attaches itself to the great, albeit now superseded, work of Peano. "Having reduced all traditional pure mathematics to the theory of the natural numbers, the next great step in logical analysis was to reduce this theory itself to the smallest set of premises and undefined terms from which it could be derived. This work was accomplished by Peano. He showed that the entire theory of the natural numbers could be derived from three primitive ideas and five primitive propositions in addition to those of

pure logic." Having paid this deserved tribute to the pioneer work of the distinguished Italian, Mr. Russell proceeds to show in detail that and how Peano's three primitive (undefined) ideas—zero, number, successor—may be defined perfectly in purely logical terms and, moreover, that and how his five primitive (undemonstrated) propositions admit of rigorous demonstration by means of purely logical concepts and propositions. This beautiful achievement, accomplished in less than 30 pages, is very weighty as showing how the entire superstructure of traditional mathematics rests upon a purely logical basis.

The natural numbers are cardinal numbers. On the threshold of this subject three questions present themselves: (1) What is meant by saying that two given classes have the same cardinal number? (2) What is meant by the cardinal number of a given class? (3) What is a cardinal number? Two classes are *similar* if they can be paired in one-one fashion like the class whose members are the legs and the class whose members are the arms of a normal human body. The answer to question (1) is: two classes have the same cardinal number if they are similar. The answer to (2) is: the cardinal number of a given class c is the class C of all classes similar to c. *i.e.*; of all classes each of which has the same cardinal number as c. The answer to (3) is: the class C' of all classes C such that each of these has for its members all the classes c similar to one of them. It is interesting to note, in passing, how far the concept, cardinal number, is removed from the domain of perception.

Evidently cardinal numbers are signless, neither positive nor negative. In order to prepare the reader for definitions (in terms of logical constants) of positive and negative integers, rational numbers, real numbers, complex numbers and other sorts, it is necessary to acquaint him with a certain serviceable stock of relations and their chief properties—relations between individuals and classes, relations subsisting between classes, and relations, such as likeness of structure, holding between relations themselves. After such preparation, which the reader receives with pleasure and a growing sense of enlightenment, he readily discerns what manner of things the various kinds of number really are. He finds, for example, that a positive or a negative integer, instead of being, like a cardinal, a class of classes, is a specific kind of *relation* between such classes of classes. More precisely, if n be any cardinal (finite or infinite) and m be a finite cardinal, the positive integer, $+ m$, is the relation of the cardinal, $n + m$, to n, and the negative integer, $- m$, is the converse relation, namely that of n to $n + m$.

What is meant by a rational number? This, too, is a relation between cardinals but a different sort of relation from the foregoing one. The direct logical approach is through the notion of propositional function. A propositional function is any statement involving one or more variables. It has the form of a proposition but is not one, being neither true nor false since the variables in it are but blanks or empty symbols. If it contain but one variable, the function determines a class of terms, namely that composed of all the terms which, if substituted for the variable, yield true propositions. If it contains two variables, say x and y, it determines a dyadic relation, which for mathematics is a class

of couples of terms, where a and b (taken in the order named) will be a couple, a member of the class, a constituent of the relation, if and only if, on being substituted, a for x and b for y, a true proposition results. We can now define precisely what is meant by the rational number m/n where m and n are definite finite cardinals. Let x and y be variables denoting finite cardinals. Then the statement $xn = ym$ is a propositional function determining a dyadic relation. The relation is a certain class of couples (a, b) of cardinals such that, if a and b be substituted respectively for x and y in the function, a true proposition results. This relation, this class of couples, is, by definition, the rational number or fraction m/n.

Such rationals are, like cardinals, signless. Positive and negative fractions are relations between the signless fractions and hence are relations of relations of cardinals.

What is a real number? There is room here for only a hint. May it suffice. A real number is neither a class of classes nor a relation of any kind used above. It is, however, a kind of relation called a *segment* of rationals. What the term means may be indicated by an example or two. The rationals arranged in order of increasing magnitude constitute a special sort of series, a special sort of relation. Now consider two parts of this relation: (1) that part consisting of all the rationals equal to or less than the rational 2; (2) that part consisting of all the rationals whose squares (rationals) are less than the rational 2. Each of these parts is a relation, an ordered class of rationals, and is called a segment of rationals. Of these segments the first has a boundary, 2, but the second has no boundary. These segments are defined to be real numbers; the former one, the real (not the rational) number 2; the latter one, the real number, $\sqrt{2}$. In case of unbounded segments it has long been customary to "postulate" a boundary and to call it a real number. It involves some labor to show that the segments themselves, which do not require to be postulated, answer the purpose. And Mr. Russell's wit is justified: "The method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil."

It is interesting to observe that in practice the same symbol is used to denote essentially different numbers. Thus, for example, the symbol, 2, is employed to denote a cardinal number, a positive integer, a rational number, a positive rational, a real number, and so on, all of which are radically different things.

For a fuller account of these matters and for a delightful excursion among kindred topics—complex and relation numbers, cardinal, ordinal and serial infinities, limits and continuity, selection theory, logical types, and so on—an interested reader must go to the book itself, and he will be amply rewarded whether he be a high school teacher of mathematics, a philosopher, or a research mathematician.

A distinguished British statesman has said: "I am a statesman but I am also a man." It is a pleasure to note, in closing this sketch, that Mr. Russell has occasionally relieved the austerity of his discourse by a brief and sudden sally, serving to show that the logical faculty, which is the human faculty *par*

excellence, consists well with the warmer elements of a broader humanity. A citation or two may be in place. "Pure logic, and pure mathematics (which is the same thing), aims at being true, in Leibnizian phraseology, in all possible worlds, not only in this higgledy-piggledy job-lot of a world in which chance has imprisoned us." Again: "It is a disgrace to the human race that it has chosen to employ the same word 'is' in two such utterances as, Socrates is human, and Socrates is a man." Once more: "Men may be defined as featherless bipeds, or as rational animals, or (more correctly) by the traits by which Swift delineates the Yahoos."

CASSIUS J. KEYSER.

COLUMBIA UNIVERSITY.

Elements of Vector Algebra. By L. SILBERSTEIN. London, Longmans, 1919. 8vo. 4 + 42 pp. Price $1.60.

Extract from the Preface—"This little book was written at the instance of Messrs. Adam Hilger, and, in accordance with their desire, it contains just what is required for the purpose of reading and handling my *Simplified Method of Tracing Rays, etc.* (Longmans, Green and Co., London, 1918). With this practical aim in view, all critical subleties have been purposely avoided. In fact, it is scarcely more than a synoptical presentation of the elements of vector algebra covering the needs of those engaged in geometrical optics. At the same time, however, it is hoped that this booklet will serve a more general purpose, viz., to provide everybody unacquainted with the subject with an easy introduction to the use of vector algebra.

"It is scarcely necessary to explain that the deductions given in this book are based on Euclid's axioms, notably with the inclusion of his postulate of parallels—upon which the equality of vectors is most essentially based. Those readers who are desirous of seeing how the formal rules here given can be generalized so as to be valid independently of the axioms of congruence and of parallels, may consult the author's *Projective Vector Algebra* (Bell and Sons, 1919), and a sequel to it published in *Phil. Mag.* for July, 1919, pp. 115–143. It is, however, advisable for the student to become first thoroughly familiar with the Euclidean vector algebra as here presented."

Contents—Section 1. Vectors defined, 1–2; 2. Equality of vectors defined, 2–3; 3. Addition of vectors, 3–10; 4. Subtraction of vectors, 10–11; 5. Scalar product of two vectors, 11–17; 6. The vector product of vectors, 17–21; 7. Expansion of vector formulæ, 21–23; 8. Iteration of vectorial multiplication 23–25; 9. The linear vector operator, 25–38; 10. Hints on differentiation of vectors, 38–40; Index, 41–42.

NOTES.

The Elementary Differential Geometry of Plane Curves by R. H. FOWLER, fellow of Trinity College, Cambridge is the latest issue, number 20, of the Cambridge Mathematical Tracts (Cambridge University Press, price 6 shillings).

The beautiful large pages, large type, and wide margins which the *American Journal of Mathematics* maintained for forty-one years have made way for pages about one-half as large, smaller type, and narrower margins.

Among the rich contents of *Archivio di Storia della Scienza,* volume 1, no. 2, June, 1919 (published February, 1920) are: an unpublished letter by Pierre Fermat, edited by G. Giovannozzi, 137–140; a bibliography of the printed works and of the facsimilies of manuscripts of Leonardo da Vinci, by A. Mieli, 177–187; a methodical bibliography of works on the history of science (326 titles) published in Italy since 1913, by A. Mieli, 195–217; and a review by G. Loria of L. C.

Karpinski's *Robert of Chester's Latin Translation of the Algebra of al Khowarismi* (New York, 1915) 218–219.

Heft 12, in December, 1919, completed the jubilee volume of *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht aller Schulgattungen*, founded by J. C. V. Hoffmann (1825–1905) in 1869.

In 1911 Teubner published the first two volumes of the great edition of Leonard Euler's *Opera Omnia*, now being prepared under the auspices of the Swiss Mathematical Society, a section of the Society of Swiss Naturalists. It was planned that the works should contain about 70 volumes and be issued in three series: series I—opera mathematica, 28 volumes; series II—opera mechanica et astronomica, 27 volumes; series III—opera physica, miscellanea, epistolae, 15 volumes. Thirteen of these volumes have been published: the two volumes of *Dioptrica* in series III; the two volumes of *Mechanica sive motus scientia analytice exposita* in series II; and the following nine volumes in series I—*Vollständige Anleitung zur Algebra;* the first of four volumes of *Commentationes arithmeticae;* the four volumes of *Institutiones calculi differentialis* and *Institutiones calculi integralis;* the first of the two volumes of *Commentationes analyticae* (integrals); and the two volumes of *Commentationes analyticae* (elliptic integrals).

In E. C. Moore's *What the War Teaches About Education and other Papers and Addresses* (New York, Macmillan, 1919) Chapter 6, pages 95–119, entitled "Does the study of mathematics train the mind specifically or universally?", was an address before the Association of Teachers of Mathematics in New England, April, 1917; chapter 7, pages 120–128, entitled "Mathematics and formal discipline again," was reprinted from *School and Society*, December 29, 1917, and is a reply to "The inadequacy of arguments against disciplinary values" by C. N. Moore; chapter 8, pages 129–151, entitled "Does the study of mathematics train the mind specifically or universally? A reply to a reply," is reprinted from *School and Society*, April 27, 1918, in reply to "Does the study of mathematics train the mind specifically or universally? A reply" by R. E. Moritz.

ARTICLES IN CURRENT PERIODICALS.

Annaes scientificas da Academia Polytechnica do Porto, volume 13, 1919, no. 2: "Pedro Nunes e os infinitamente pequenos" by R. Guimaraes, 65–71—No. 3: "Sur les surfaces réglées" by C. Servais, 129–151; "Sur l'octaëdre à faces triangulaires" by J. Neuberg, 161–171.

Bulletin de la Société Mathématique de France, volume 47, nos. 1–2, 1919: "Les fondements de la théorie des fonctions elliptiques" by H. Hancock, 37–42.

Bulletin des Sciences Mathématiques, volume 54, August, 1919: Review by H. Deltheil of G. Castelnuovo's *Calcolo delle Probabilità* (Milano-Roma-Napoli, 1919), 165–174; Review by E. Cartan of G. Bouligand's *Cours de géométrie analytique* (Paris, 1919), 175–178; "Sur les intégrales de Fresnel" by A. Bloch, 179–180.

Harvard Graduates Magazine, volume 28, March 1920: "An American liaison officer in Paris" by J. L. Coolidge, 394–408.

Mathematical Gazette, volume 9, no. 143, December, 1919: "Report of the Mathematical Association committee on the teaching of mathematics in public and secondary schools,"

393–421; "Gleanings far and near," 421; Note on "The series for sin x and cos x" by F. Jackson, 422–424; "Books received, contents of journals, etc.," i–iv; "The teachers' library," i–viii [section III (history, about 60 titles), section IV (biography, about 40 titles), section V (philosophy, about 35 titles)]—Volume 10, no. 144, January, 1920: "The relations of geography and mathematics in a well-equipped secondary school" by J. Fairgrieve, 1–2; "The graphical treatment of differential equations" (continued) by S. Brodetsky, 3–8; "Gleanings far and near," 8, 13; "Ratio and proportion" by D. K. Picken, 9–13; "Note on Napier's logarithms" by W. R. Bower, 14–16.

Messenger of Mathematics, volume 49, no. 1, May, 1919: "Factorisation of N & $N' = (x^n \mp y^n) \div (x \mp y)$, &c. [when $x - y = 1$]," by A. Cunningham, 1–16—June: [*same*], 17–32.

Nature, volume 104, December 25, 1919: "The reform of the calendar" by H. C. P[lummer], 415–416 [First sentence: "Those who have concerned themselves with the question of a reformed calendar will find much interesting matter in a report published by a committee which was appointed early in the year by the Paris Société d'Encouragement pour l'Industrie nationale]—January 1, 1920: "Deflection of light during a solar eclipse" by A. Anderson, 436; "Einstein's theory and a map analogue" by E. Cunningham, 437—January 8: "The neglected study of probabilities" by G. S. Bryan, 464–465 [Review of Castelnuovo's book *Calcolo delle Probabilità*]; (Milano-Roma-Napoli, 1919).

Quotation: ["Among the sins of omission for which mathematicians and teachers of mathematics might be charged, there is probably none which has so vitally affected our national welfare as the neglect of the study of probabilities. Into every event of ordinary life considerations of probability enter in a greater or less degree, and for this reason every boy or girl who has learnt to use vulgar fractions ought to be taught to apply them to simple games of chance, and in this way to illustrate the rules for fractional addition, multiplication, and so forth. In default of this knowledge, millions of pounds are spent on postal orders in response to attractive offers of hundred-pound prizes the expectation value of which is not one-tenth of the price paid. Yet it was only recently that the Central Welsh Board excluded probabilities from an examination syllabus in algebra which was simply loaded up with questions on collections of letters and symbols that could convey no meaning to the victims of the examination system.

"Other applications are to such problems as life assurance and statistics . . . Prof. Castelnuovo's treatise strikes the reviewer as just the kind of book of which it would be worth while to publish an English translation. . . . In our universities a great deal of waste energy in the departments of pure mathematics could also be utilized by turning out graduates with a knowledge of probabilities and statistics which would filter down through the teachers to the elementary schools and thus to the citizens of the future."] "Deflection of light during a solar eclipse" by A. Schuster, 468; "Relativity and radio-activity" by J. Joly, 468; "Retirement of Sir Oliver Lodge," 483—February 26: "A mathematician's miscellany," review by N. R. C. of M. Lecat's *Pensées sur la science, la guerre et sur des sujets très variés* (Bruxelles, 1919), 689–690; "Gravitational deflection of high-speed particles" by Leigh Page, 692–693—Volume 105, March 4: "The Einstein deflection of light" by A. D. Crommelin, 23–24—March 11: "Mathematical cosmogony" by H. C. Plummer, 31–32 [Review of J. H. Jean's *Problems of cosmogony and stellar dynamics*, (Cambridge, 1919)]; "Gravitational deflection of high-speed particles" by A. S. Eddington, 37; "Gravitational shift of spectral lines" by H. Jeffreys, 37–38; "The mariner's compass," 44; "The gyrostatic compass" by S. G. Brown, 44–48.

Nouvelles Annales de Mathématiques, volume 78, August, 1919: "Théorème général sur les équations algébriques" by M. Petrovitch, 281–284; "Sur l'identité de Begout" by B. Gambier, 284–297; "Concours d'agrégation de 1914," 297–311; "Solutions de questions proposés," 311–320—September: "Démonstration du théorème fondamental de la théorie des équations algébriques" by L. Pomey, 321–324; "Sur le rapport anharmonique" by M. Auric, 325–329; "Sur les courbes à axe orthoptique et les courbes de direction" by M. d'Ocagne, 329–338; "Remarques géométriques sur une question de concours à l'Ecole Polytechnique en 1919" by P. du Plessis, 338–342; "Agrégation de l'enseignement secondaire des jeunes filles: concours de 1919. —Problème de géométrie et de géométrie analytique," 342–354; "Sur la détermination du rayon de courbure en coordonnées polaires" by M. d'Ocagne, 354–356; "L'enseignement mathématiques à l'université de Strasbourg," 356–360—October: "Sur le complexe de Painvin," by A. Myller, 361–365; "Quelques applications d'une remarque de M. d'Ocagne" by M. F. Egan, 365–368; "Quelques applications des formules vectorielles (L)" (continued) by—Levengle, 368–373; "Trois

démonstrations des théorèmes de Fermat et de Wilson" by L. Pomey; 373–380 ["Si p est un nombre premier et a un entier non divisible par lui, p divise $a^{p-1} - 1$ (Fermat) et $(p-1)! + 1$ (Wilson)"]; "Enveloppe des plans des faces des hexaèdres dont les diagonales sont portées par des droites données" by G. Fontené, 380–391; "Observations sur les triangles rectangles en nombres entiers et les suites de Fibonacci" by C. A. Laisant, 391–397; Review by R. Bricard of G. Bouligand's *Cours de géométrie analytique* (Paris, 1919), 397–399; "Questions," 399–400.

PHILOSOPHICAL MAGAZINE, London, sixth series, volume 38, December, 1919: "The kinematics of the eye" by H. Lamb, 685–695 ["This subject should be of interest to mathematicians, for it is almost the only field where the theory of finite rotations, so elegant geometrically, finds a practical application.

"The theory of the movements of the eye was investigated very completely by Helmholtz,[1] and forms a considerable chapter of his Physiologische Optik.[2] The mathematical part of the exposition is rather elaborate, and has I think somewhat obscured the essential simplicity of the matter. There is a decided gain in this respect, as I wish to show, if we adopt a purely geometrical treatment, based mainly on theorems of Donkin and Hamilton, which were indeed anterior in date to Helmholtz's work but could hardly have been known to him"]; "On some aspects of the theory of probability" by Dorothy Wrinch and H. Jeffreys, 715–731 ["Summary: It is shown that the attempt to give a definition of probability in terms of frequency is unsuccessful. Laplace's definition, apparently in these terms, really involves implicitly the concept of probability and is therefore circular in character. Veun's definition in terms of the limit of a series is unsatisfactory because there is no reason to believe that his series do in fact usually tend to a limit; it is shown that there are many cases where they do not; and as his process is incapable of being carried out, the existence of such a limit can in any case only be known *a priori* if at all, so that his method offers no advantage over that of regarding probability as an entity known to exist independently of definition, intelligible without such definition, and perhaps indefinable.

"A set of axioms on which a mathematical theory of probability can be based is then given, which seems to offer certain advantages over the current ones. In particular it is capable of covering cases where the principle of sufficient reason cannot be applied to assess probability. It is also shown that a complete theory of probability must allow for the use of infinitesimals.

"A discussion of sampling induction is given, in which it is shown that when the sample is large enough the prior probabilities of different constitutions of the whole do not usually affect appreciably the probabilities inferred after the samples have been taken. Also the range within which the fractional constitution is as likely as not to lie includes the fractional constitution of the sample, and its extent is inversely proportional to the number of the sample itself, whatever be the number of the whole."]—January, 1920: "Note on the construction of a parabolic trajectory and a property of the parabola used by Archimedes" by W. B. Morton and T. C. Tobin, 157–160.—February: "The recent eclipse results and Stokes-Planck's aether" by L. Silberstein, 161–170; "Note on the possible structure for the aether" by O. J. Lodge, 170–174.—March: "Parametric solutions for a fundamental equation in the general theory of relativity, with a note on similar equations in dynamics" by E. T. Bell, 285–288; "On the measurement of time—a rejoinder to Dr. N. Campbell" by L. Silberstein, 366–372; Review of J. H. Jeans's *Problems of Cosmogony and Stellar Dynamics* (Cambridge, 1919), 377–378; Review of L. C. Karpinski, H. Y. Benedict and J. W. Calhoun's *Unified Mathematics* (Boston, 1918), 380–381 [Last paragraph: "The whole book is very elegant and stimulating, and carries Perry's pioneering ideas to a high stage of development and pitch of perfection"]; Review of H. Lamb's *Elementary Course of Infinitesimal Calculus* (Revised edition, Cambridge, 1919), 381–383.

PROCEEDINGS OF THE EDINBURGH MATHEMATICAL SOCIETY, volume 37 (session 1918–19), issued November, 1919: "The Stirling numbers and polynomials" by C. Tweedie, 2–25; "Note on the complete Jacobian elliptic integrals" by F. Bowman, 26–32; "A theorem of Sonine in Bessel functions, with two extensions to spherical harmonics" by J. Dougall, 33–47; "On the stereometric generation of the De Jonquières transformation" by J. F. Tinto, 48–58; "The cosymmedian system of tetrahedra inscribed in a sphere" by W. L. Marr, 59–64; "Determinantal systems of apolar triads in a conic" by W. L. Marr, 65–68; "Determinantal systems of apolar triads in the twisted cubic" by W. L. Marr, 69–75; "Approximations to the length of an arc" by D. M. Y. Sommerville, 76–79; "Relations between the integrals of the hypergeometric equa-

[1] "Ueber die normalen Bewegung des menschlichen Auges" (1863), reprinted in *Wissenschaftliche Abhandlungen*, vol. 2, p. 360.

[2] First edition, 1866; second edition, 1896.

tion" by T. M. MacRobert, 80–84; "Theorems connected with the differentiation of a circulant" by W. H. Metzler, 85–96; "List of Members, 1918–19" 97–104.

PROCEEDINGS OF THE ROYAL SOCIETY, London, volume 97, no. A681, March 1, 1920: "Discussion on the theory of relativity" [on February 5, 1920] by J. H. Jeans, A. S. Eddington, F. W. Dyson, A. Fowler, E. Cunningham, H. F. Newall, and L. Silberstin, 66–79.

REVISTA MATEMÁTICA HISPANO-AMERICANA, volume 1, nos. 8–9, October–November, 1919: "Daniel Augusto da Silva, su vida y su obra cientifica" by F. Gomes Teixeira, 237–243; "Puntos notables del triángulo" by P. M. González Quijano, 244–251; "Algunas nociones de análisis situs" by O. Fernández Baños, 252–258; "La sucesión de Fibonacci" (third article) by F. Vera, 259–263; Problems and solutions, 264–269; "Crónica," "Notas necrológicas," "Revista de revistas," "Bibliografia," "Glosario matemático," 270–292; "Volumen del prismatoide" by L. Catalá, 293–296.

SCHOOL REVIEW, Chicago, volume 28, no. 1, January ,1920: "A failure class in algebra" by O. A. Wood, 41–49.

SCHOOL AND SOCIETY, volume 10, December 27, 1919: "The work of the national committee on mathematical requirements," 765–766; "A queer mathematical conjecture" by G. A. Miller, 773—Volume 11, January 24, 1920: "The work of the national committee on mathematical requirements" by J. W. Y[oung], 110–111.

SCHOOL SCIENCE AND MATHEMATICS, volume 19, no. 9, December, 1919: "What is practical mathematics?" by D. H. Moskowitz, 827–829; "Marginal notes on Cajori's History of Mathematics" by G. A. Miller, 830–835; "Some mathematical howlers," 835; Problem department, 836–840; "The socialized recitation in mathematics" by F. E. Buss, 844–848; Book reviews, 862–868—Volume 20, no. 1, January, 1920: "A few live projects in high school mathematics" by F. M. Rich, 34–45; "Applied mathematics for high schools" by E. H. Barker, 46–51; "Determination of the Brocard points" by J. C. Camplain, 51; Problem department, 86–90; "Notes and news regarding the work of the national committee on mathematical requirements," 92.—February: "The relation of vocational guidance to our teaching of science and mathematics" by Anna Y. Reed, 105–113; "Correlation of mathematical subjects" by E. R. Brestich, 125–134; "Weighting data" by J. P. Ballantine, 140–141; "The differential pulley or chain hoist" by W. F. Roecker, 142–143; Problems and solutions, 174–176.

SCIENCE, new series, volume 51, January 2, 1920: "Time, space and gravitation" by A. Einstein, 8–10 [Quotation: "Thus the new theory of gravitation diverges widely from that of Newton with respect to its basal principle. But in practical application the two agree so closely that it has been difficult to find cases in which the actual differences could be subjected to observation. . . . No one must think that Newton's great creation can be overthrown in any real sense by this or by any other theory. His clear and wide ideas will for ever retain their significance as the foundation on which our modern conceptions of physics have been built."]—January 9: "Weight of a body moving along equator" by E. V. Huntington, 45–46 ["To the Editor of *Science*: A prominent engineer, Dr. Carl Hering, recently proposed to me the following question: 'Does a body in motion along the earth's equator weightless (or more) than the same body at rest?' Since this question, in some form or other, has come up several times in recent discussions, the following solution, although entirely elementary, may be not without interest. . . . There are two speeds [of a train], one westward of about 18,700 miles per hour, and one eastward of about 16,700 miles per hour, at which the 'weight' of the body as measured by an observer on the train (that is, the tension in the supporting string . . .) would be zero."]; "An odd problem in mechanics" by C. Hering, 46 [the one discussed by E. V. Huntington]—January 16: "Recent advances in dynamics" by G. D. Birkhoff, 51–55 [address of the vice-president and chairman of Section A, Mathematics and Astronomy, of the American Association for the Advancement of Science, St. Louis, December, 1919]—January 23: "Concerning application of the probable error in cases of extremely asymmetrical frequency curves" by E. L. Michael, 89–91; "The American Mathematical Society" by F. N. Cole, 91–92—February 6: "Gravity and aerostatic pressure on fast ships and air-planes," by A. McAdie, 144–145—February 13: "Further history of the calculus" by A. S. Hathaway, 166–167—February 20: "The conference at Cleveland on the history of science" by L. Thorndike, 193–194—February 27: "A proposed method for carrying triangulation across wide gaps" by H. L. Cooke and H. N. Russell, 211–213; "Two new base maps of the United States," 213–214—March 5: "Some aspects of physics in war and peace" by G. F. Hull, 221–233 [Address of the vice-president and chairman of Section B, Physics, American Association for the Advancement of Science, St. Louis, December, 1919]; "Mathematics at the University of Stras-

bourg" by E. B. Wilson, 243—March 12: "Einstein's law of gravitation" by J. S. Ames, 253–261 [Presidential address delivered at the St. Louis meeting of the Physical Society, December 30, 1919]; "An odd problem in mechanics" by W. D. Lambert, 271–272 [First paragraph: "The following statements are intended to throw light on the questions raised by Dr. Hering in his letter entitled 'An odd problem in mechanics' in *Science* for January, 9, 1920"]—March 26: "Modern interpretation of differentials" by E. V. Huntington, 320–321 ["Without attempting to discuss the historical questions involved, I wish to point out that the theory of 'differentials' given by Professor A. S. Hathaway in *Science* for February 13, would prove highly misleading to the modern student"]; "Weight and centripetal acceleration" by B. L. Newkirk, 321 [First sentence: "Mr. Carl Hering's suggestion for a new form of dynamic compass ought to be challenged before some one organizes a company to work the idea out on a commercial basis."]

SCIENCE PROGRESS, volume 14, January, 1920: "Recent advances in pure mathematics" by Dorothy Wrinch, 361–369; "Evolution and irreversibility" by A. J. Lotka, 406–417; "Philip Edward Bertrand Jourdain," 448–451; Reviews by Dorothy Wrinch of L. E. Dickson's *History of the Theory of Numbers: Vol. I, Divisibility and Primality* (Washington, 1919) and of C. E. Cullis's *Matrices and Determinoids*, Vol. II (Cambridge, 1918), and by P. E. B. Jourdain of J. Milne's *The Analytical Geometry of the Straight Line and the Circle* (London, 1919), 497–499.

TECHNOLOGY REVIEW, Cambridge, Mass., volume 22, no. 1, January, 1920: [Various tributes to Richard Cockburn Maclaurin, 1870–1920, portrait frontispiece], 13–49.

TÔHOKU MATHEMATICAL JOURNAL, volume 16, nos. 3–4, November, 1919: "Flat-sphere geometry," (third paper) by J. Eiesland, 184–235; "On a certain class of geometrical figures" by K. Yoneyama, 236–263; "On the relations between the roots of certain two transcendental functions" by Y. Okada, 264–269; "On the theory of Stäckel curvature" by K. Ogura, 270–290; "On the convergence-abscissa of a certain definite integral" by K. Kurosu, 291–298; "On the *wansanists* of the Sendai, Ichnoseki, Morioka, and Hirosaki *hans*" [in Japanese] by T. Hayashi, 299–333; New Books, Contents of Periodicals, Miscellaneous Notes, 334–348.

TRANSACTIONS OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, volume 22, no. 15, October, 1919: "Asymptotic satellites near the equilateral-triangle equilibrium points in the problem of three bodies" by D. Buchanan, 309–340.

UNIVERSITY OF CALIFORNIA PUBLICATIONS IN MATHEMATICS, volume 1, no. 8, February 18, 1920: "A list of Oughtred's mathematical symbols with historical notes" by F. Cajori, 171–186—No. 9, February 17: "On the history of Gunter's scale and the slide rule during the seventeenth century" by F. Cajori, 187–209—No. 10, February 17: "On a birational transformation connected with a pencil of cubics" by A. R. Williams, 211–222—No. 11, February 17: "Classification of involutory cubic space transformations" by F. R. Morris, 223–240.

UNIVERSITY OF TEXAS BULLETIN, no. 1905, January 20, 1919: *Some dangers in establishing a pension system and the proper precautions* by E. L. Dodd, 26 pp.

WASHINGTON UNIVERSITY STUDIES, scientific series, volume 7, no. 2, January, 1920: "Notes on the geometry of the characteristic curves of the synchronous motor" by A. S. Langford, 47–57; "Geometric description of the halo on the dome of the St. Louis cathedral" by W. H. Roever, 59–65 [An analytic derivation of the form of this halo was given by the author in this MONTHLY, December, 1913, volume 20, pp. 299–303].

YALE REVIEW, volume 9, no. 3, April, 1920: "Einstein's theory of gravitation" by L. Page, 570–585.

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, volume 50, 1919, nos. 6–7: "Die Interpolation im mathematischen Unterricht" by G. Wolff, 201–210; "Geduldspiele" by B. Kerst, 211–212; "Konstruktion des Schussnetzes für die Artillerie" by A. Krebs, 213–217; "Graphische Lösung der Gleichung $x^3 + ax + b = 0$" by O. Dörge, 217–219; "Über die Verknüpfung des Pothenotschen Problems mit einigen anderen" by F. Redl, 219–222; "Das Buchsieb" by Friese, 223–225; "Zeichnerische Lösung astronomischer Aufgaben" by W. Misar, 225–227; "Das Minimum der Ablenkung" by M. Brües, 227–228—Nos. 8–9: "Der Einfluss der Kriegs erfahrungen auf den mathematisch-physikalischen Unterricht der höheren Schulen" by W. Lietzmann, 249–267; "Ueber zwei Fragen, die mit dem Problem der Referenz-flächen des Himmels und der Gestirne zusammenhängen" by A. Müller, 267–271—No. 10: "Rudolf Sturm zum Gedächtnis" (with portrait) by W. Lorey, 289–293; "Ein Beitrag zur schulgemässen Behandlung der Trägheits-momente" by E. Rehfeld, 294–301; "Zum Satz von Busche" by R. Mehmke, 301–305; "Modell zur Anwendung der Peripheriewinkelsätze" by A. Finzel, 305–307.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

NOTES AND SUGGESTIONS.

A Junior Mathematics Club was organized at Cornell University in the early part of the present academic year. One feature of this club's work seems to the editor well worth incorporating into the program of other undergraduate clubs. In response to a request from the students, a member of the faculty devoted an hour to explaining to the club "the utility and value of each of the mathematical courses offered by the University beyond the freshman year; what courses would be of greatest value to the student of chemistry, of physics, of biology, of economics, of education, and for further mathematical development; and, in each case, some of the reasons why the course suggested was of particular value in the field indicated."

In response to a request published some time ago for suggestions as to mathematical games suitable for club uses in social meetings, Mrs. W. E. BECKWITH of the College for Women, Western Reserve University, Cleveland, Ohio, sends the following: "Place a fairly good-sized mirror upright on a table and have each student draw a four-inch square with its diagonals by watching his movements in the mirror. I find it necessary to cover the hands to insure use of the mirror only. This undertaking is far more difficult than it would appear to be and is certain to 'break the ice' on any occasion."

In response to the same request Professor ELIZABETH B. COWLEY of Vassar College, Poughkeepsie, N. Y., writes that their club has used mathematical adaptations of some of the familiar guessing games. For example, a theorem in geometry was selected and each person of a group took one word of the theorem. The person who had been sent "out" returned and asked each a question to which she replied, using in the reply, her word in the theorem. This game was most successful when the theorem selected was quite familiar. Another game was that in which one member was sent out and all the others selected a mathematical term. When the person returned, she had the privilege of asking twenty questions about the term, selecting the persons to be questioned. Each person questioned must answer, but without mentioning the term. Of course, if the questions are skilful and the answers are less so, the term is quickly guessed. At one of its meetings the club played charades using mathematical terms as the words to be represented by acts.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, Providence, R. I. [*1918*, 33–34; *1919*, 167; *1920*, 28.]

The Mathematics Club of Brown University issues annually a printed folder announcing the program for the year and containing the list of officers and

information concerning dates and places of meeting. From the 1919–20 folder we have taken the data given below.

The programs for the year are as follows:

November 7, 1919: "Women as mathematicians and astronomers" by Rachel T. Easterbrooks '20; "Geometric proofs of the law of tangents" by Clarence M. Eddy '22; "Logarithmic spiral shapes for cams and wheels" by Chauncey D. Wentworth '20.

December 12: "Hyperbolic functions and some of their practical applications" by Fred L. Itschener '21; "Probabilities in the game of shooting craps" by Everett L. Sweet '21.

February 13, 1920: "History and development of logarithms" by Allen A. Edgcomb '22; "Arithmetic prodigies" by Harold B. Yeaton '21.

March 12: "Imaginary quantities and their graphs" by Martha W. Watt, Gr.; "Map-making" by Raymond L. Wilder '20.

April 9: "Projective methods of generating curves" by Professor Helen A. Merrill of Wellesley College, Wellesley, Mass. Club picture.

May: (date to be announced later): Picnic.

At some time during the year Professor David Eugene Smith of Columbia University, N. Y., is to give an illustrated stereopticon lecture on "The history of mathematics" but at the time of this writing (April) the date has not yet been definitely determined.

THE JUNIOR MATHEMATICAL CLUB, University of Wisconsin, Madison, Wis.
[*1918*, 188–189, 457–458.]

Regular meetings of the club are held on the second and fourth Thursdays of each month. It is planned to have faculty members speak at about half of the meetings and students (usually two at each meeting) furnish the programs for the other meetings. The club has two picnics each year—one in the fall for the purpose of arousing interest in the club and of getting the mathematics students acquainted with each other, and one late in the spring which serves as a farewell to the senior members.

The club is open to both men and women and includes not only those majoring in mathematics but also a number who have minors in mathematics (chiefly students of physics and chemistry). The membership for the year 1918–19 was 21 and for the year 1919–20 between 25 and 30.

The officers for the year 1918–19 were: President, Eva Thornton '19; vice-president, Arlisle Schumacher '21; secretary-treasurer, Ruth Conlin '19.

Because of the S. A. T. C. and the influenza epidemic no meetings were held in the fall of 1918. The programs for the meetings since then up to the time of sending in this copy are given below.

January 21, 1919: "French mathematics" by Professor Edward B. Van Vleck.

February 4: "Italian mathematicians" by Professor L. W. Dowling.

February 13: Continuation of discussion of "Italian mathematicians" including some personal experiences as a student in Italy, by Professor Dowling.

February 27: "Who's who in modern mathematics" by Professor Ernest B. Skinner.
March 13: "The trisection of an angle by means of conic sections" by Madge Ryan '20; "Curves" by Ethel Vasey '19.
April 16: Social evening at the home of Professor and Mrs. Dowling. Illustrated lecture on "Famous mathematicians" by Professor Dowling.
May 8: "Works of Archimedes" by Professor Charles S. Slichter.
May 22: Business meeting for election of officers. Officers elected for the year 1919–20: President, Margaret Lee '20; vice-president, Ruth-Marie Urban '20; secretary-treasurer, Gladys Baur '20.
October 21: Picnic.
October 30: "Classification of curves and surfaces" by Professor Van Vleck.
November 13: "Some surprises in the history of mathematics" by Professor Dowling.
December 4: "Mathematical recreations" by Estelle Stone '20 and Madge Ryan '20.
December 18: "Graphical calculation" by Professor Arnold Dresden.
January 8, 1920: "Numerals of antiquity" by Ruth-Marie Urban '20; "Life and works of Abel" by Alta Gudsos '20.
January 22: "Integers" by Professor Skinner.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about problems and solutions to B. F. FINKEL, Springfield, Mo.

NOTE ON CAUSTICS.

By OTTO DUNKEL, Washington University.

The purpose of this note is to show that the formula (3), which was derived by Professor da Cunha in another way in his solution of problem 2768, below, has a somewhat wider application. If from a point F rays of light fall upon the concave side of a curve Γ in a plane containing the point, and the reflected rays intersect on the same side, the envelope γ of these reflected rays is called the caustic of Γ with respect to the point F. Let M be a point of Γ, F' the point in which the ray reflected from M has contact with γ, ω the angle of incidence, $R = MO$, the radius of curvature of Γ at M, $\delta = FM$ and $\delta' = MF'$. Then it may be proved that

$$\frac{1}{\delta} + \frac{1}{\delta'} = \frac{2}{R\cos\omega}. \tag{1}$$

A simple geometrical derivation of this result is given in Humbert's *Cours d'Analyse*, vol. 1, page 77. Let the point of intersection of the normal to Γ at M with the line FF' be denoted by T and the length MT by n. By dropping

perpendiculars from F and F' upon the normal MT and by considering two pairs of similar triangles, it is easily seen from a figure that

$$\frac{n - \delta \cos \omega}{\delta' \cos \omega - n} = \frac{\delta}{\delta'} \quad \text{or} \quad \frac{1}{\delta} + \frac{1}{\delta'} = \frac{2 \cos \omega}{n}. \tag{2}$$

From (1) and (2) there results at once

$$R = \frac{n}{\cos^2 \omega}, \tag{3}$$

the formula mentioned above.

If the curvature of the curve Γ is such that the point F' lies on the side opposite to F, then by a very slight modification of the derivation given by Humbert it may be shown that (1) is again true provided that δ' be regarded as negative. In this case there is no caustic by reflection in the physical sense and it is not considered by Humbert. Also a proof similar to that indicated above shows that (2) also is true if δ' be regarded as negative. Thus (3) is true for both cases. It is thus somewhat more general and at the same time simpler than (1); it also has the advantage of being more convenient for geometrical constructions. Thus, if the point O has been found on the evolute of a curve Γ corresponding to the point M of Γ, the formula (3) gives an easy construction for the point F' on the caustic of Γ with respect to a given point F. If, for example, the curve Γ is a circle the caustic may be easily traced by this construction. If, on the other hand, Γ is a conic and F is a focus, then γ reduces to the other focus, or to a point at infinity in the case of a parabola, and the formula gives the construction given by Professor da Cunha.

Another construction for the center of curvature may be derived from a property of triangles. In any triangle FMF' let C be the middle point of the side FF', T the point in which the bisector of the angle at M meets FF', P the point in which the side FM is cut by the perpendicular to the bisector at T, O the point in which the perpendicular to FM at P meets the bisector MT (O is the center of curvature when F, F' and M have the meaning above) and finally Q the point in which TP meets CM. The property referred to is the fact that QO is perpendicular to FF'. A similar theorem is true for the external bisector. The second construction, which applies to the general case as well as to conics, is then as follows: Erect a perpendicular to the normal at the point where it cuts the axis and from the point in which this perpendicular meets the line joining the center with the given point on the curve drop a perpendicular to the axis and produce it to meet the normal. This point is the center of curvature.

PROBLEMS FOR SOLUTION.

2829. Proposed by E. S. PALMER, New Haven, Conn.

Given a set of arbitrary pairs of positive integers (a_p, b_p), $(p = 1, 2, \cdots, n)$: (*a*). Is it always possible to find a set of positive integers k_p, $(p = 1, 2, \cdots, n)$ such that

$$k_p a_p + k_p b_p > \sum_{r=1}^{r=n} k_r a_r, \qquad (p = 1, 2, 3, \cdots, n).$$

(*b*) If or when possible, show how to find k_p?

2830. Proposed by WILLIAM HOOVER, Columbus, Ohio.

An elastic string connects a pair of opposite vertices of a square whose sides are four equal rods freely jointed, each of length $2a$. The system is suspended vertically from a vertex attached to the string and is at rest. If the string be cut, and θ is the acute angle any side makes with the vertical at any moment during the motion, determine the angular velocity of any rod.

2831. Proposed by B. J. BROWN, Kansas City, Mo.

Given that the series $\sum_0^\infty a_n y^n$ is absolutely convergent when $|y| < 1$, prove that the series $\sum_0^\infty a_n(2x\cos\theta - x^2)^n$ may be arranged in powers of x provided $|x| < \frac{2}{3}$.

Prove that if x lies between 1 and 2, the series $\sum_0^\infty (2x - x^2)^n$ is convergent and consists only of positive terms, but that the series obtained by arranging it in powers of x, diverges.

2832. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that the square of the sum of four squares is the sum of four squares, that the square of the sum of eight squares is the sum of six squares, and that the square of the sum of sixteen squares is the sum of ten squares.

2833. Proposed by W. H. ECHOLS, University of Virginia.

In *Engineering*, London, during 1917 appeared the following equations concerning the stability of ships; they are employed by the naval constructors in the Norfolk, Virginia, Navy Yard, and they are of importance,

$$\tan^3\theta_0 + \frac{2m}{\rho}\tan\theta_0 - \frac{2x_0}{\rho} = 0,$$

$$\tan^3(\theta_0 + \theta_1) + \frac{2m}{\rho}\tan(\theta_0 + \theta_1) - \frac{2x_0}{\rho} - \frac{2x}{\rho} = 0,$$

$$\tan^3(\theta_0 - \theta_2) + \frac{2m}{\rho}\tan(\theta_0 - \theta_2) - \frac{2x_0}{\rho} + \frac{2x}{\rho} = 0.$$

The required unknowns are θ_0, x_0 and m. The constants have values as follows, θ_1 and θ_2 are positive angles ranging from 15′ to 10°, x is positive and less than 5, and ρ is positive with considerable range of values. A rapid solution involving small labor is desired, determining x_0 within the same limits as given for x.

SOLUTIONS OF PROBLEMS.

2740 [1919, 35]. Proposed by E. W. CHITTENDEN, University of Illinois.

Establish the identity (the determinant is of order n):

$$\begin{vmatrix} a-\lambda, & a, & a, & \cdots, & a \\ a, & a-\lambda, & a, & \cdots, & a \\ a, & a, & a-\lambda, & \cdots, & a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a, & a, & a, & \cdots, & a-\lambda \end{vmatrix} = (-1)^{n-1}\lambda^{n-1}(na - \lambda).$$

SOLUTION BY L. E. MENSENKAMP, Freeport, Illinois.

We proceed to establish this relation by mathematical induction. The relation is obviously true for $n = 2$. Assume it is true for a determinant of order n as written above by the proposer. Multiply both sides of this equality by $\frac{-\lambda(na + a - \lambda)}{na - \lambda}$. The right-hand member of the

resulting expression is $(-1)^n\lambda^n[(n+1)a-\lambda]$, and is of the form demanded by our theorem by the case of $n+1$. The left-hand member of the resulting expression may be written so that its determinant is of order $n+1$,

$$\frac{na+a-\lambda}{na-\lambda}\cdot\begin{vmatrix} -\lambda, & 0, & 0, & \cdots, & 0 \\ a, & a-\lambda, & a, & \cdots, & a \\ a, & a, & a-\lambda, & \cdots, & a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a, & a, & a, & \cdots, & a-\lambda \end{vmatrix} \qquad (1)$$

Here the $-\lambda$ of our multiplier appears as the element in the first row and first column of the determinant of order $n+1$. Add the second, third, and each of the following rows in turn to the first row obtaining an equal expression, the new determinant being exactly the same as that in (1) above except that the elements of the first row are now all $na-\lambda$. Multiply the factor preceding this determinant into the elements of the first row obtaining a third determinant which is again exactly like (1) except that the elements of the first row are now all $na+a-\lambda$. Subtract the second, third, and each of the following rows from the first, thus obtaining a determinant of order $n+1$ and of precisely the form demanded by our theorem if it is to be true in the case of $n+1$. Thus, if the relation is true for n, it is also true for $n+1$, and the induction is complete.

Also solved by A. L. CANDY, P. J. DA CUNHA, L. D. HAND, A. M. HARDING, R. A. JOHNSON, L. C. MATHEWSON, H. L. OLSON, A. PELLETIER, J. L. RILEY, G. Y. SOSNOW, ELIJAH SWIFT, and C. C. YEN.

2744 [1919, 37]. Proposed by E. B. ESCOTT, Chicago, Ill.

An insurance company computes its quarterly premiums by adding 6 per cent to the annual premium and dividing by 4. If a policyholder pays quarterly, what rate of interest is he paying?

I. Solution by Elijah Swift, University of Vermont.

If we assume that this means that the policyholder sets aside the annual premium at the beginning of the year, pays the first of the quarterly premiums out of it, lets the remainder lie at interest for three months, then deducts the second premium, and so on, the interest will be compounded quarterly, and the present worth of the four premiums at the beginning of the year must equal the annual premium. If we call the annual premium, $4P$, and the (unknown) annual interest rate, $4i$, each quarterly premium will be $1.06P$ and we have the equation

$$1.06P+\frac{1.06P}{1+i}+\frac{1.06P}{(1+i)^2}+\frac{1.06P}{(1+i)^3}=4P.$$

This cubic may be solved by Horner's method, whence $4i = 16.11$ per cent.

If interest be compounded semi-annually, we have a quadratic,

$$1.06P+\frac{1.06P}{1+i}+\frac{1.06P}{1+2i}+\frac{1.06P}{(1+i)(1+2i)}=4P$$

whence $4i = 16.33$ per cent.

If interest be reckoned as simple (compounded annually) we have to solve the cubic

$$1.06P+\frac{1.06P}{1+i}+\frac{1.06P}{1+2i}+\frac{1.06P}{1+3i}=4P.$$

whence $4i = 16.54$ per cent.

In any case, then, the policyholder must pay over 16 per cent for the accommodation.

II. Solution by the Proposer.

A (by algebra).

Let P = annual premium, p = quarterly premium $= \frac{1.06}{4}P$, r = annual rate of interest.

$$P=\frac{400}{106}p.$$

We have the equation

$$(P - p)\left[1 + \frac{r}{4}\right]^3 - p\left[1 + \frac{r}{4}\right]^2 - p\left[1 + \frac{r}{4}\right] - p = 0.$$

Putting

$$1 + \frac{r}{4} = x$$

and substituting value of P, we have the equation

$$294x^3 - 106x^2 - 106x - 106 = 0.$$

Solving by Horner's Method, we have

$$x = 1 + \frac{r}{4} = 1.04028,$$

$$r = .16112 = 16.112 \text{ per cent compounded quarterly.}$$

B (by arithmetic). A more elementary and more "practical" method is the method by trial and error. A few trials will show that the rate is something over 16 per cent.

First Trial. Taking the rate as 16 per cent and the annual premium as 100, we have the scheme,—

Annual premium due	100.00
First quarterly premium paid	26.50
	73.50
Interest for three months	2.94
	76.44
Second quarterly premium paid	26.50
	49.94
Interest for 3 months	2.00
	51.94
Third quarterly premium paid	26.50
	25.44
Interest for 3 months	1.02
	26.46
Fourth quarterly premium	26.50
First error	0.04

We see that 16 per cent is slightly too small.

Second trial. Taking the rate as 16.2 per cent., we have, in the same way as before, an error of + .03.

Forming a table

Rate	Error
16	−.04
16.2	+.03

By interpolation, the rate that will give zero error is

$$16 + \frac{4}{7} \times .2 = 16.114 \text{ per cent.}$$

If greater accuracy were required, repeat the computation with the last rate and interpolate again.

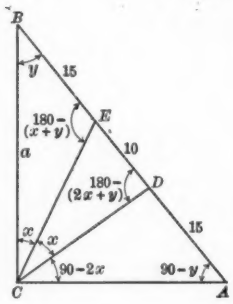
Also solved by G. N. ARMSTRONG, H. N. CARLETON, and H. L. OLSON.

2747 [1919, 72]. Proposed by DANIEL KRETH, Wellman, Iowa.

In the right triangle ABC, right angle C, we have given on the hypotenuse the segments $AD = 15$, $DE = 10$, $EB = 15$, and the angle DCE equal to the angle ECB. Find the angle DCE, and the sides AC and BC.

SOLUTION BY MARCIA LATHAM, New York City.

Let $BC = a$, $AC = b$, $DC = c$, $\angle DCE = \angle ECB = x$, $\angle ABC = y$. Then $\angle DCA = 90 - 2x$, $\angle EDC = 180 - (2x + y)$, and $\angle BEC = 180 - (x + y)$.



$$\sin y = b/40 \quad \text{and} \quad \cos y = a/40. \tag{1}$$

Since EC bisects $\angle BCD$,

$$a/c = 15/10 = 3/2. \tag{2}$$

In the triangle, DCA, by the law of sines,

$$c/15 = \cos y/\cos 2x. \tag{3}$$

Combining (1), (2), and (3), $2a/45 = a/40 \cos 2x$, or $\cos 2x = 9/16$, $x = 1/2 \cos^{-1} 9/16 = 27° 53' 8''$. In the triangle, BCD, by law of sines, $\sin (2x + y)/\sin y = a/c$. Whence, by (1) and (2)

$$\sin (2x + y) = 3b/80. \tag{4}$$

Also, $a/25 = \sin (2x + y)/\sin 2x$; whence, by (4), $\sin 2x = 15b/16a$. But $\sin^2 2x + \cos^2 2x = 1$. Then $(15b/16a)^2 + (9/16)^2 = 1$; whence

$$b^2 = 7a^2/9. \tag{5}$$

Now, in the triangle, ABC, $a^2 + b^2 = (40)^2$; whence, by (5), $a^2 + 7a^2/9 = 1{,}600$; whence, $a = 30$ and from (5), $b = 10\sqrt{7}$.

Also solved by A. M. HARDING, POLYCARP HANSEN, C. E. HORNE, R. A. JOHNSON, ELMER LATSHAW, E. W. MARTIN, LOUIS ORDANKSY, A. PELLETIER, J. L. RILEY, H. M. ROESER, L. SMITH, D. L. STAMY, H. TSAI, and L. G. WELD.

2760 [1919, 124]. Proposed by CHARLES N. SCHMALL, New York City.

In an arithmetical progression, if s_n be the sum of the first n terms, s_{2n} the sum of the first $2n$ terms, and s_{3n} the sum of the first $3n$ terms of the same series, prove that $s_{2n} - s_n = \frac{1}{3}s_{3n}$.

SOLUTION BY EMMA M. GIBSON, Springfield (Mo.) High School.

The sum of n terms of an arithmetical progression is expressed by the formula

$$s_n = \frac{n(a_1 + a_n)}{2},$$

where a_1 and a_n are the first and nth terms, respectively.

Hence, $s_n = n(a_1 + a_n)/2$, $s_{2n} = 2n(a_1 + a_{2n})/2$, and $s_{3n} = 3n(a_1 + a_{3n})/2$ are the sums of the first n terms, the first $2n$ terms, and the first $3n$ terms, respectively. Now $a_{2n} = a_n + nd$, $a_{3n} = a_{2n} + nd = a_n + 2nd$, d being the common difference.

Then $s_{2n} = 2n(a_1 + a_n + nd)/2$ and $s_{3n} = 3n(a_1 + a_n + 2nd)/2$ and

$$s_{2n} - s_n = 2n(a_1 + a_n + nd)/2 - n(a_1 + a_n)/2 = n(a_1 + a_n + 2nd)/2 = s_{3n}/3.$$

Also solved by R. D. BOHANNAN, H. L. BRIDGES, JR., H. N. CARLETON, W. F. CHENEY, JR., P. J. DA CUNHA, H. C. GOSSARD, WILLIAM HERBERG, C. N. MILLS, LOUIS O'SHAUGHNESSEY, H. L. OLSON, A. PELLETIER, J. B. REYNOLDS, I. S. SUN, and ELIJAH SWIFT.

2768 [1919, 171]. Proposed by PAUL CAPRON, U. S. Naval Academy.

Given the center, a focus, and a point of a conic, construct geometrically the circle of curvature at the point.

I. Solution by P. J. da Cunha, University of Lisbon.

Soient F le foyer donné, C le centre, F' l'autre foyer, determiné sur FC par $CF' = FC = c$, $2a$ et $2b$ les axes de la conique, $e = \sqrt{a^2 \pm b^2}/a$ son excentricité, $2p$ son paramètre, M le point donné, δ et δ' les rayons vecteurs FM et $F'M$, R et n le rayon de courbure et la longueur de la normale MT à la conique au même point, 2ω l'angle FMF' pour l'ellipse et son supplément pour l' hyperbole, x, y les coordonnées de M, en supposant la conique rapportée à ses axes.

En considérant d'abord le cas d'ellipse, nous partirons des formules connues

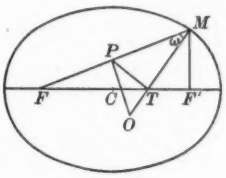
$$(1) \qquad R = \frac{n^3}{p^2}, \qquad n = \frac{1}{a^2}\sqrt{a^4y^2 + b^4x^2}, \qquad \delta = a + ex, \qquad \delta' = a - ex,$$

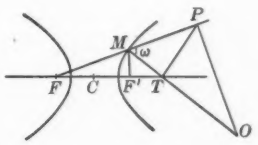
d'où il résulte, en premier lieu

$$n^2 = \frac{a^4y^2 + b^4x^2}{a^4} = \frac{a^2(a^2b^2 - b^2x^2) + b^4x^2}{a^4} = \frac{a^4b^2 - b^2(a^2 - b^2)x^2}{a^4} = \frac{b^2(a^2 - e^2x^2)}{a^2} = \frac{b^2\delta\delta'}{a^2}$$

ou

$$(2) \qquad \delta\delta' = \frac{a^2}{b^2}n^2.$$





Le triangle FMF' donne, d'autre part, $4c^2 = \delta^2 + \delta'^2 - 2\delta\delta' \cos 2\omega = (\delta + \delta')^2 - 4\delta\delta' \cos^2 \omega$ ou, en égard à l'équation (2) $4c^2 = 4a^2 - (4a^2/b^2)n^2 \cos^2 \omega$, d' où il vient, toutes réductions faites, $n \cos \omega = b^2/a = p$. La première des formules (1) donne ensuite

$$(3) \qquad R = \frac{n}{\cos^2 \omega}.$$

Un calcul analogue, appliqué à l'hyperbole, nous conduirait au même résultat. Or la formule (3) résout le problème, parce qu' elle donne immédiatement, la construction suivante:

Par le point T, où la normale coupe l'axe FF', on mène une perpendiculaire à MT, et par le point P, ou elle coupe FM, on mène une autre perpendiculaire à FM. Le point de rencontre O de cette droite avec MT est le centre de courbure demandé.

Obs. L'énoncé du problème suppose une conique à centre, mais il est facile de vérifier que le résultat obtenu s'applique également à la parabole.

Note. La solution ci-dessus n'est pas nouvelle. Voir Duhamel, *Eléments de calcul infinitesimal*, tome 1, Paris, 1856, p. 172–173.

II. Solution by Grace M. Bareis, Ohio State University.

Let the center, focus and point of the conic be designated by C, F and M, respectively. On the line through CF, locate F', the other focus, so that $F'C = CF$. Describe the circle through F', F, M. Join M to the ends of the diameter that passes through C. These two lines are the tangent and normal to the conic at M. The conic is a hyperbola or an ellipse according as the line that divides the segment CF internally or the one that does not so divide it is taken as tangent. The problem, therefore, has two solutions.

Let L and T be the points in which the tangent and normal, respectively, cut the line joining C and F. Let L' be the foot of the perpendicular from M on CF; then, L' and L are conjugate points with respect to the conic. Now find A and A', the double points in the involution determined by the pair of points L and L' and the center of the involution C. These points A and A' are the vertices of the conic. The lines through A and A', respectively, and perpendicular to CF

are the tangents at these points. The conic is, therefore, determined; for we have three points M, A, A' and the tangent at each of them, known.

From this point on the construction may be completed in any one of three ways:

(1) By the use of Pascal's Theorem, find the point H in which the normal at M cuts the conic again. Draw the circle on HM as diameter. Find the fourth point of intersection G of this circle with the conic. Draw HG. Through M draw MK parallel to HG and determine the point K in which this line cuts the conic again. The circle tangent to the conic at M and passing through K is the required circle of curvature at M. (Cf. L. Cremona, *Elements of Projective Geometry*, Oxford, 1885, p. 190).

(2) By the method previously described both the ellipse and the hyperbola that have center C, focus F, and point M are determined. The construction may, therefore, be completed by making use of the fact that at the intersection of two confocal conics the center of curvature of either is the pole with respect to the other of the tangent to the former at the intersection.

(3) Through F, draw a line parallel to the tangent at M. By Desargues' Theorem, find the points R and R' in which this line meets the conic. On MF take Q so that $MQ = RR'$. Then the circle through Q and tangent to the conic is the required circle of curvature; for, the focal chord of curvature is equal to the focal chord of the conic drawn parallel to the tangent at the point. For results in (2) and (3) the reader may consult Salmon's *Conic Sections*, 6th ed., 1879, pp. 374–376.

It may be noted that if C is at infinity, the conic becomes a parabola with the axis CF, the focus F, and the point M given. Two parabolas may be drawn satisfying these conditions according to which of the bisectors of the angle made by MF and the diameter through M is taken as tangent. The above methods of construction then apply to this case also.

Also solved by H. HALPERIN, A. PELLETIER, J. B. REYNOLDS, and the Proposer.

2771 [1919, 191]. Proposed by GEORGE PAASWELL, New York City.

A circle is revolved through an angle of 90° about a vertical chord which does not pass through the center of the circle. Taking the origin at the lower extremity of the chord, the z-axis along the chord and the x- and y-axes in the boundary planes, pass a plane through the x-axis making a given angle with the xy plane. Determine the portion of the area of the surface above the plane and between the xz and yz planes.

I. SOLUTION BY J. B. REYNOLDS, Lehigh University.

A solution by vector analysis. Let $r = a\cos\theta + b\sin\theta$ be the polar equation of the circle revolved. From this we see that for a vector equation of the surface generated, we may write $r = (a\cos t + b\sin t)\cos t\cos u\cdot i + (a\cos t + b\sin t)\cos t\sin u\cdot j + (a\cos t + b\sin t)\sin t\cdot k$; or

$$r = \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\cos u\cdot i + \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\sin u\cdot j + \tfrac{1}{2}\{b + a\sin 2t - b\cos 2t\}\cdot k$$

from which

$$\frac{\partial r}{\partial t} = \{-a\sin 2t + b\cos 2t\}\cos u\cdot i + \{-a\sin 2t + b\cos 2t\}\sin u\cdot j + \{a\cos 2t + b\sin 2t\}\cdot k,$$

$$\frac{\partial r}{\partial u} = -\tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\sin u\cdot i + \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\cos u\cdot j,$$

giving the vector product

$$\frac{\partial r}{\partial t}\times\frac{\partial r}{\partial u} = -\tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\{a\cos 2t + b\sin 2t\}\cos u\cdot i - \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\{a\cos 2t + b\sin 2t\}\sin u\cdot j + \tfrac{1}{2}\{-a\sin 2t + b\cos 2t\}\{a + a\cos 2t + b\sin^2 2t\}\cdot k.$$

[1] Of course it is unnecessary to describe the circle through F', F, M, for the tangent and normal bisect angles given at M; nor is it necessary to invoke an involution in order to get A and A' since we have given $F'M + FM = 2a$.—EDITOR.

Now since a differential element of the surface generated is the absolute value of $\frac{\partial r}{\partial t} \times \frac{\partial r}{\partial u}$ multiplied by $dr\,du$, we have

$$S = \int\int \left[\frac{\partial r}{dt} \times \frac{\partial r}{\partial u}\right]_0 du dt$$

$$= \frac{\sqrt{a^2+b^2}}{2} \int\int \{a + a\cos 2t + b \sin 2t\} dt du.$$

In order to get the limits of this integral, we must find the value of t for which the plane $r = xi + yj + y\tan\alpha \cdot k$, the plane through the x-axis making an angle α with the xy plane, cuts the surface of revolution. Equating coefficients of i, j, and k in the two equations, we find

$$x = \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\cos u, \qquad y = \tfrac{1}{2}\{a + a\cos 2t + b\sin 2t\}\sin u$$

$$y\tan\alpha = \tfrac{1}{2}\{b + a\sin 2t - b\cos 2t\},$$

the latter two of which give us the equation

$$b\cot\alpha - a\sin u + (a\cot\alpha - b\sin u)\sin 2t - (b\cot\alpha + a\sin u)\cos 2t = 0;$$

whence, $t = \tan^{-1}(-a/b)$ or $\tan^{-1}(\sin u/\cot\alpha)$ and since the first of these two values of t is for the special point where $r = 0$, we have

$$S = \frac{\sqrt{a^2+b^2}}{4} \int_0^{\pi/2} du \{2at + a\sin 2t - b\cos 2t\}_{\tan^{-1}\frac{\sin u}{\cot\alpha}}^{\pi/2}$$

$$= \frac{\sqrt{a^2+b^2}}{2} \int_0^{\pi/2} \left\{\frac{\pi a}{2} - a\frac{\cot\alpha\sin u}{\csc^2\alpha - \cos^2 u} + b\frac{\cot^2\alpha\csc^2 u}{\csc^2\alpha + \cot^2\alpha\cot^2 u} - a\tan^{-1}\frac{\sin u}{\cot\alpha}\right\} du$$

$$= \frac{\sqrt{a^2+b^2}}{2}\left\{\frac{\pi^2 a}{4} + \frac{\pi b\cos\alpha}{2} + a\frac{\cos\alpha}{2}\log\left[\frac{1-\sin\alpha}{1+\sin\alpha}\right]\right\} - a\frac{\sqrt{a^2+b^2}}{2}\int_0^{\pi/2}\tan^{-1}\left(\frac{\sin u}{\cot\alpha}\right) du$$

for the required surface. If $\alpha = \pi/2$, $S = 0$. If $a = 0$, $S = \pi b^2/4\cos\alpha$, one fourth the surface of a sphere of diameter b less half a zone of height $b/2(1-\cos\alpha)$. If $\alpha = 0$ and $b = 0$, $S = \pi^2 a^2/8$ one eighth the surface of a torus. If $\alpha = 0$,

$$S = \frac{\pi^2 a\sqrt{a^2+b^2}}{8} + \frac{\pi b\sqrt{a^2+b^2}}{4},$$

as it should, since a semicircle of length $\frac{\pi}{2}\sqrt{a^2+b^2}$ is revolved through 90° about an axis at a distance $a/2 + b/\pi$ from its center of gravity. Since for $\alpha < 45°$, $\tan^{-1}(\sin u/\cot\alpha)$ will expand into an absolutely convergent series we may use for such values of α to any desired degree of accuracy

$$\int_0^{\pi/2}\tan^{-1}\left(\frac{\sin u}{\cot\alpha}\right) du = \sum_{n=1}^{n=\infty} (-1)^{n+1}(\tan\alpha)^{2n-1}\frac{2\cdot 4\cdot 6\cdots 2n-2}{3\cdot 5\cdot 7\cdots 2n-1}\frac{1}{2n-1}.$$

II. Solution by B. F. Finkel, Drury College.

Those of our readers who are unfamiliar with quaternions or in whose hands this subject is not an easily operated instrument of investigation can obtain the integral S by the following process:

Using the same notation; that is, taking r, t, and u as polar coördinates, we have for the equation of the surface of revolution $r = a\cos t + b\sin t$, and for the element of arc on circle revolved $ds = \sqrt{a^2+b^2}\cdot dt$.

The required area is expressed by the double integral

$$S = \int\int r\cos t\, du\, ds$$

$$= \sqrt{a^2+b^2}\int\int (a\cos^2 t + b\sin t\cos t)\, du\, dt.$$

The limits for u are 0 and $\pi/2$, and as the equation of the given plane is $\tan t = \tan \alpha \sin u$, the limits for t will be $\tan^{-1}(\tan \alpha \sin u)$ and $\pi/2$.

Therefore

$$S = \frac{\sqrt{a^2+b^2}}{2}\int_0^{\pi/2} du \int_{\tan^{-1}(\tan \alpha \sin u)}^{\pi/2} (a + a\cos 2t + b \sin 2t)\, dt.$$

2772 [1919, 171]. Proposed by HARRY LANGMAN, New York City.

Given $1 = (-\frac{1}{2} + x)^r = (-\frac{1}{2} - x)^r$, where r is integral. Prove that r is a multiple of 3. In general, if

$$1 = \left[\cos\frac{2\pi}{m} + x\right]^r = \left[\cos\frac{2\pi}{m} - x\right]^r,$$

where r and m are integers, prove that r is a multiple of m.[1]

SOLUTION BY THE PROPOSER.

The general case only will be considered, and the trivial case $r = 0$ excluded.

Put

$$a = \cos\frac{2\pi}{m} + x \quad \text{and} \quad b = \cos\frac{2\pi}{m} - x. \tag{1}$$

Then

$$a + b = 2\cos\frac{2\pi}{m}. \tag{2}$$

We may write

$$1 = \cos 2k\pi + i \sin 2k\pi. \tag{3}$$

Hence,

$$a = \cos\frac{2k\pi}{r} + i\sin\frac{2k\pi}{r} \quad \text{and} \quad b = \cos\frac{2k'\pi}{r} + i\sin\frac{2k'\pi}{r},\ 0 < k, k' < r+1. \tag{4}$$

From (2),

$$\cos\frac{2k\pi}{r} + \cos\frac{2k'\pi}{r} + i\left(\sin\frac{2k\pi}{r} + \sin\frac{2k'\pi}{r}\right) = 2\cos\frac{2\pi}{m},$$

from which

$$\cos(k+k')\frac{\pi}{r}\cdot\cos(k-k')\frac{\pi}{r} = \cos\frac{2\pi}{m} \quad \text{and} \quad \sin(k+k')\frac{\pi}{r}\cdot\cos(k-k')\frac{\pi}{r} = 0. \tag{5}$$

Hence, we must have

$$\sin(k+k')\frac{\pi}{r} = 0. \tag{6}$$

From the problem, if $m \neq 1$, we must have $a \neq b$. Hence, in (4) $k \neq k'$. Hence, we must have $k + k' < 2r$, that is, $(k+k')/r < 2$. But, from (6), $(k+k')/r$ must be integral. Hence, since $k + k' > 0$,

$$k + k' = r \quad \text{and} \quad k - k' = 2k - r. \tag{7}$$

From the first equation of (5), we then obtain (8) $\cos\frac{2\pi}{m} = -\cos\frac{2k-r}{r}\pi$.

Now $\frac{2}{m} \leqq 1$ and $\left|\frac{2k-r}{r}\right| \leqq 1$. Therefore, from (8),

$$\left|\frac{2k-r}{r}\right| = 1 - \frac{2}{m}.$$

Hence, $\frac{2k}{r} - 1 = 1 - \frac{2}{m}$ or $1 - \frac{2k}{r} = 1 - \frac{2}{m}$, from which $k = r - \frac{r}{m}$ or $k = \frac{r}{m}$. From (7), $k' = \frac{r}{m}$ or $k' = r - \frac{r}{m}$. Since k and k' are integral, we must have $\frac{r}{m}$ integral. Hence, r is a multiple of m.

[1] If $m = 4$ we may take $x = 1$, $r = 2$; in this case r is not a multiple of m. Therefore the heorem of the question is not true for this case.—EDITORS.

Also solved by C. A. BARNHART, H. HALPERIN, H. L. OLSON, and A. PELLETIER.

2773 [1919, 212]. Proposed by JOSEPH ROSENBAUM, Milford, Conn.

Point out the fallacy in the proof of the following problem:

In the triangle $A_1B_1C_1$ let m be a point such that the sum of the distances from it to the sides is a maximum; also let $A_2B_2C_2$ be a triangle formed by drawing lines through the vertices A_1, B_1, and C_1 parallel to their opposite sides. Then the sum of the distances from m to the sides of the triangle $A_2B_2C_2$ is a minimum.

Proof.—Because the sides of the two triangles are parallel in pairs, the sum of the distances from a variable point P in triangle $A_1B_1C_1$ to the six sides of the two triangles is constant. Now by hypothesis M is a point for which one part of this constant sum is a maximum, and hence it follows that the other part is a minimum.

SOLUTION BY H. L. OLSON, University of Wisconsin.

This proof is correct, with the understanding that if a point P is on the opposite side of BC, for example, to the vertex A, the distance to the side BC is to be regarded as negative. It is easy to see, however, that the point M does not exist, and that the proposition is therefore vacuous. Represent the perpendicular distances from P to the sides BC, AC, and AB by α, β, and γ respectively. If we denote by Δ the area of the triangle ABC, we are to minimize the function $\alpha + \beta + \gamma$, subject to the condition $a\alpha + b\beta + c\gamma = 2\Delta$. ($a$, b, and c represent, as is customary, the sides BC, AC, and AB, respectively.) Eliminating γ, we have, as the function to be minimized,

$$\left(1 - \frac{a}{c}\right)\alpha + \left(1 - \frac{b}{c}\right)\beta + \frac{2\Delta}{c}.$$

Hence, the derivatives, $\left(1 - \frac{a}{c}\right)$, and $\left(1 - \frac{b}{c}\right)$, of this function with respect to α and β must vanish; but for the general triangle they do not vanish and hence M does not exist. If, however, $a = b = c$, the sum of the distances is the constant $2\Delta/c$; likewise the sum of the distances for the corresponding triangle $A_2B_2C_2$ is constant.

Also solved by A. PELLETIER and A. L. WECHSLER.

2774 [1919, 212]. Proposed by FRANK IRWIN, University of California.

Evaluate the circulants

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 3 & 4 & \cdots & n & 1 \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \end{vmatrix},$$

where, in the latter, $a_1, a_2, \cdots, a_n$ form an arithmetical progression.

I. SOLUTION BY P. J. DA CUNHA, University of Lisbon, Portugal.

Denote the first of these circulants by Δ and the second by Δ'. Let

$$s_n = \frac{1+n}{2}n$$

be the sum of the first n positive integers. Add to the elements of the last line of Δ the sum of the corresponding elements of all the preceding lines. We obtain a determinant which we can write as the product

$$\Delta = s_n \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ n & 1 & 2 & \cdots & n-3 & n-2 & n-1 \\ n-1 & n & 1 & \cdots & n-4 & n-3 & n-2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 4 & 5 & 6 & \cdots & 1 & 2 & 3 \\ 3 & 4 & 5 & \cdots & n & 1 & 2 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{vmatrix}.$$

If we subtract from the elements in the first $n-1$ columns of this determinant the corresponding elements of the last column, we shall obtain a determinant which we can easily reduce to the order $n-1$. Now subtract from the elements of the first $n-2$ lines of this new determinant the corresponding elements of the last line. We obtain after a little reduction

$$\Delta = -s_n \begin{vmatrix} -n & -n & -n & \cdots & -n \\ 0 & -n & -n & \cdots & -n \\ 0 & 0 & -n & \cdots & -n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & -n \end{vmatrix} = (-n)^{n-1}\frac{1+n}{2},$$

the determinant being of order $n-2$.

Pass now to the determinant Δ'. Putting

$$a_k = a_1 + (k-1)r \quad (k = 1, 2, 3, \cdots, n),$$

we shall have

$$\Delta' = \begin{vmatrix} a_1 & a_1+r & a_1+2r & \cdots & a_1+(n-2)r & a_1+(n-1)r \\ a_1+(n-1)r & a_1 & a_1+r & \cdots & a_1+(n-3)r & a_1+(n-2)r \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ a_1+r & a_1+2r & a_1+3r & \cdots & a_1+(n-1)r & a_1 \end{vmatrix}.$$

The same reasoning will lead us to

$$\Delta' = (-n)^{n-1}r^{n-1}\left(a_1 + \frac{n-1}{2}r\right),$$

as it is easy to see.

II. Solution by Otto Dunkel, Washington University.

The reduction of these special circulants can be made to depend upon the known fact that the general circulant reduces to the product $f(\omega_1)f(\omega_2)\cdots f(\omega_n)$, where the ω's are the roots of $x^n - 1 = 0$ and $f(x) = a_1 + a_2x + a_3x^2 + \cdots + a_nx^{n-1}$ (see Cesàro, *Elementares Lehrbuch der algebraischen Analysis* etc., page 25). If we set $a_k = a_1 + (k-1)r$, then

$$f(\omega) = a_1[1 + \omega + \omega^2 + \cdots + \omega^{n-1}] + r[\omega + 2\omega^2 + 3\omega^3 + \cdots + (n-1)\omega^{n-1}],$$

$$= 0 + \frac{rn}{\omega - 1}, \text{ if } \omega \neq 1,$$

$$= a_1n + \frac{r(n-1)n}{2} = n\left[a_1 + \frac{n-1}{2}r\right], \text{ if } \omega = \omega_1 = 1,$$

whence

$$\Delta' = \frac{r^{n-1}n^n\left[a_1 + \frac{n-1}{2}r\right]}{(\omega_2 - 1)(\omega_3 - 1)\cdots(\omega_n - 1)} = (-1)^{n-1}r^{n-1}n^{n-1}\left[a_1 + \frac{n-1}{2}r\right].$$

Also solved by H. L. Olson and A. Pelletier.

2777 [1919, 268]. Proposed by W. D. Cairns, Oberlin College.

Prove that the two series

$$1 + \frac{\pi^4}{2^4 \cdot 4!} + \frac{\pi^8}{2^8 \cdot 8!} + \cdots$$

and

$$\frac{\pi^2}{2^2 \cdot 2!} + \frac{\pi^6}{2^6 \cdot 6!} + \frac{\pi^{10}}{2^{10} \cdot 10!} + \cdots$$

are equal.

Solution by H. S. Uhler, Yale University.

By substituting x, $-x$, ix, and $-ix$ for y in the absolutely convergent series

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^n}{n!} + \cdots$$

it will be found at once that

$$\tfrac{1}{4}[(e^x + e^{-x}) + (e^{ix} + e^{-ix})] = 1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \cdots + \frac{x^{4n}}{(4n)!} + \cdots, \tag{1}$$

$$\tfrac{1}{4}[(e^x + e^{-x}) - (e^{ix} + e^{-ix})] = \frac{x^2}{2!} + \frac{x^6}{6!} + \frac{x^{10}}{10!} + \cdots + \frac{x^{4n-2}}{(4n-2)!} + \cdots, \tag{2}$$

where $n = 1, 2, 3, \cdots$.

Again, since $e^{i\theta} = \cos\theta + i \sin\theta$ it follows that $e^{i(\pi/2)} + e^{-i(\pi/2)} = i - i = 0$.

Consequently by substituting $\pi/2$ for x in formulas (1) and (2) we see at a glance that each of the given series has the same limit $\frac{1}{4}(e^{\pi/2} + e^{-\pi/2})$, that is, the series are "equal."

Also solved by W. W. BEMAN, P. J. DA CUNHA, A. M. HARDING, H. L. OLSON, A. PELLETIER, S. W. REAVES, ELIJAH SWIFT, E. H. WORTHINGTON, and the Proposer.

2785 [1919, 366]. Proposed by W. H. ECHOLS, University of Virginia.

If on the sides, as bases, of any closed plane polygon, there be constructed similar triangles similarly placed, all outward or all inward, then the centroid of the vertices of these triangles coincides with the centroid of the corners of the polygon.

SOLUTION BY THE PROPOSER.

Let $Z_1, \cdots, Z_n \equiv Z_1$ be the n corners of the polygon, the Z's being complex numbers. The sides of the polygon are respectively

$$\Delta Z_r \equiv Z_{r+1} - Z_r, \qquad (r = 1, \cdots, n-1)$$

and $\Sigma \Delta Z_r = 0$, since the polygon is closed.

The n vertices of the similar triangles constructed similarly on the sides are

$$w_r = Z_r + k\Delta Z_r \cdot e^{i\alpha}, \qquad (r = 1, \cdots, n-1)$$

k being a real constant factor and α a real constant angle.

Hence,

$$\Sigma w_r = \Sigma Z_r + ke^{i\alpha}\Sigma\Delta Z_r$$

and therefore,

$$\frac{1}{n}\Sigma w_r = \frac{1}{n}\Sigma Z_r.$$

Also solved by S. W. REAVES and ELIJAH SWIFT.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

At Ohio State University, Messrs. VAN B. TEACH, V. B. CARIS and D. L. HOLL have been assistants in mathematics for the present year.

H. R. BRAHANA, of Princeton University, has been appointed instructor in mathematics at the University of Illinois for 1920–1921.

Miss MAY J. SPERRY, of Brown University, has been appointed instructor in mathematics and physics, at Knox College, Galesburg, Ill., for 1920–21.

At Brown University Associate Professor H. P. MANNING has resigned after twenty nine consecutive years of service as a teacher in the department of mathematics. Dr. R. F. BORDEN, of the University of Illinois, has been appointed instructor, and Messr sC. D. WENTWORTH and R. L. WILDER graduate assistants, for 1920–21.

At Cornell University, Dr. H. C. M. MORSE, of Harvard University, and Mr. P. A. FRALEIGH, of Dartmouth College, have been appointed instructors in mathematics for 1920–21.

Dr. C. C. CAMP, of the University of Illinois, has been appointed assistant professor of mathematics at Iowa State College.

Dr. CHESTER SNOW has resigned as professor of mathematics at the University of Idaho to accept a position as physicist in the Bureau of Standards, Washington, D. C.

According to *Science*, Professor A. W. BUTTERFIELD, of the department of mathematics at the Worcester Polytechnic Institute, has resigned to become educational director for the Norton Company.

Dr. C. N. REYNOLDS, of Wesleyan University, has been appointed instructor at Dartmouth College for 1920–21.

Dr. O. D. KELLOGG, who has been lecturer in mathematics at Harvard University during the past year, has been promoted to be professor of mathematics.

Miss MARIAN M. TORREY, of Brown University, who has been teaching for two years in the Phebe Anna Thorne Model School of Bryn Mawr College, has been appointed instructor in mathematics at the University of West Virginia.

At the University of Nancy Dr. LÉOPOLD LEAU (*1920*, 89) has been appointed professor of differential and integral calculus to replace Dr. A. S. E. HUSSON.

At the University of Lyons, Dr. L. SIRE, maître de conférence at the University of Rennes has been appointed professor of applied mathematics in place of the late Professor D. J. B. FLAMME.

At the University of Lille, Dr. A. CHATELET has been appointed professor of general mathematics.

Mr. W. E. H. BERWICK has been appointed lecturer in mathematics in the University of Leeds. *Nature* states:

"Mr. Berwick was assistant lecturer in the University of Bristol for two years, and afterwards became lecturer in mathematics in University College, Bangor. For two years he was engaged on the staff of the anti-aircraft experimental section of the Munitions Inventions Department at Portsmouth, where he made important contributions to the experimental and computative theory of gunnery. He has published a long series of papers in the *Proceedings of the London Mathematical Society* and elsewhere."

Professor G. E. FISHER, of the department of mathematics in the University of Pennsylvania, died March 28, 1920 within about three weeks of his fifty-seventh birthday. He was instructor in mathematics at Cornell University 1887–89, assistant professor at the University of Pennsylvania 1889–1908, and professor since 1908. He was a charter member of the ASSOCIATION.

With Professor I. J. SCHWATT he collaborated in issuing the authorized translation of the fourth German edition of H. Durège's *Elements of the Theory of Functions* (Philadelphia, 1896); *Text-book of Algebra*, part 1 (Philadelphia, 1898); *Rudiments of Algebra*, 1899; *School Algebra*, 1899; and other algebras.

Professor G. D. BIRKHOFF, of Harvard University, has been elected a member of the Royal Danish Academy of Sciences.

Professor L. E. DICKSON, of the University of Chicago, has been elected a member of the American Philosophical Society.

Mr. J. H. JEANS, of Dorking, England, formerly professor of applied mathematics at Princeton University, has been nominated as secretary of the Royal Society of London.

Professor D. R. CURTISS, of Northwestern University, gave an address before the Schoolmasters Club at Ann Arbor, April 3, on "Mathematics."

Professor E. W. BROWN delivered a lecture on the history of mathematics before the Gamma Alpha fraternity of Yale University on February 26. The Astronomical Society of the Pacific has awarded to Professor Brown its Bruce Medal for 1920.

The Silliman lectures at Yale University for 1920 were delivered by Professor JACQUES HADAMARD, of the Collège de France, on April 30, May 3 and May 5. The general subject of the lectures was "Some topics in linear partial differential equations."

At the meeting of the National Research Council on April 28, the Division of Physical Sciences voted to increase its membership by adding a representative of the MATHEMATICAL ASSOCIATION OF AMERICA, and a tenth member-at-large, who should represent mathematics. There were previously in the division fourteen physicists, three representatives of the American Mathematical Society (L. E. DICKSON, O. VEBLEN, and H. S. WHITE), three representatives of the American Astronomical Society, one meteorologist and one geodesist.

The fourteenth regular meeting of the American Mathematical Society was held at the University of Chicago, April 9–10. Nineteen papers were presented. At the symposium (*1920*, 144) Professor MAX MASON discoursed on "The electromagnetic field equations" and Professor A. C. LUNN on "The theory of relativity." There was an attendance of over a hundred at the meeting. At the

dinner on the evening of April 9 Professor L. E. DICKSON reported recommendations of the Committee on Bibliography of which Professor R. C. ARCHIBALD is chairman. On the presentation of a report, including these recommendations, to the Council in New York City on April 23, it was resolved that the Council is of the opinion that a journal of mathematical abstracts is very desirable, and the Committee was authorized to take steps toward securing the financial aid necessary to found such a journal and was requested to present to the Council plans for its organization. The plan for the establishment of a journal of mathematical abstracts was also heartily approved on April 28 by the Division of Physical Sciences of the National Research Council.

At the annual meeting of the National Academy of Sciences in Washington April 26–28 the following papers were read by members of the Association: By A. G. WEBSTER, (a) "The Springfield rifle and the heduc formulæ"; (b) "Some new methods in internal ballistics of the Springfield rifle"; (c) "Preliminary measurements on the pressures in the 'onde de choe'"; (d) "On the connection of the specific heats with the equation of state of a gas"; by L. E. DICKSON, "Recent notable progress in the theory of numbers"; by E. KASNER, "Geodesics and relativity."

Professor M. E. C. JORDAN of the Collège de France was elected a foreign associate of the Academy, and Professor H. F. BLICHFELDT, of Stanford University, the thirteenth member in the Section of Mathematics. The other twelve members are: G. D. BIRKHOFF, G. A. BLISS, O. BOLZA, L. E. DICKSON, E. KASNER, E. H. MOORE, W. F. OSGOOD, W. E. STORY, E. B. VAN VLECK, O. VEBLEN, H. S. WHITE, and E. T. WILCZYNSKI.

In the American Association for the Advancement of Science what has been prior to 1920, Section A, Mathematics and Astronomy, has been divided into two Sections: Section A, Mathematics, and Section B, Astronomy. The officers of Section A are as follows—Vice-president, D. R. CURTISS; Secretary, W. H. ROEVER; Members of Sectional Committee: 5 years, D. JACKSON; 4 years, A. D. PITCHER; 3 years, G. A. BLISS; 2 years, J. PAGE; 1 year, H. L. RIETZ; Member of the Council, G. A. MILLER; Member of the General Committee, E. V. HUNTINGTON. In Section B the Vice-president is J. STEBBINS, and the Secretary F. R. MOULTON.

At the meeting of the Association of Teachers of Mathematics in New England at Springfield, Mass., March 13, 1920, the following papers by college teachers were presented: by Professor J. W. YOUNG, Dartmouth College, "The work of the National Committee on Mathematical Requirements and the National Council of Teachers of Mathematics"; by Professor ELEANOR C. DOAK, Mount Holyoke College, "Inscriptible polygons"; by Mr. J. S. MIKESH, Yale University, "Courses for teachers of mathematics in secondary schools." At the spring meeting of the Association, at Boston University, May 1, the following papers

were read by members of the Association: By L. R. PERKINS, "College courses for teachers of mathematics"; by C. L. E. MOORE, "Einstein's theory."

At the meeting of Mathematics Teachers of New Jersey in Elizabeth, N. J., April 17, 1920, addresses were delivered by the following members of the ASSOCIATION: By F. DURELL, "The organization of graphic methods" (presidential address); by O. VEBLEN, "Displacements and symmetries in three dimensions"; by H. E. WEBB, (*a*) "Note on the pure quadratic equation," (*b*) "Note on the proof of Euler's theorem $e^{i\theta} = \cos\theta + i\sin\theta$," and (*c*) "Certain questions arising from the report of the National Committee."

A new organization, to be known as the National Council of Teachers of Mathematics, was launched at Cleveland, Ohio, on February 24. About 150 persons were present, representing 20 different states and many different organizations of primary and secondary school teachers in various parts of the country. The following officers were elected—President, C. M. Austin, Oak Park, Ill.; vice-president, H. O. Rugg, New York, N. Y.; secretary-treasurer, J. A. Foberg, Chicago, Ill.; executive committee: Marie Gugle, Columbus, O. (3 years); J. Rorer, Philadelphia, Pa. (3 years); H. Wheeler, Worcester, Mass. (2 years); W. D. Reeve, Minneapolis, Minn. (1 year). Two members are to be appointed by the committee itself.

Mathematics Teacher, in reorganized form, will probably be the official organ of the Council, administered by an editorial board of from three to five members and an editor in chief. This board is to consist of teachers of elementary and secondary mathematics, and to include a member representing the college group in an advisory capacity.

Sir THOMAS MUIR has recently deeded 2500 books from his mathematical library to the South African Public Library at Cape Town. This collection is mainly made up of serial publications and writings bearing on the theory of determinants, and allied matters. So far as is known the library is in this latter respect the most complete in existence.

Additional announcements (cf. *1920*, 192–194) of mathematics courses in Summer Sessions are as follows:

Leland Stanford University, June 22–September 3. By Professor H. F. BLICHFELDT, Columbus, 4 hrs.; Coördinate geometry, 4 hrs.; Advanced course. By H. W. BRINKMAN, Trigonometry, 5 hrs.; Algebra, 4 hrs.

University of Missouri, June 21–August 14. By Professor L. INGOLD, Second calculus and Seminar. By Mr. E. ALLEN, Trigonometry and Analytic geometry.

Ohio State University, June 21–August 13. By Professor H. W. KUHN, Fundamental concepts of algebra and geometry, 4 hrs.; Modern higher algebra, 3 hrs. By Professor S. E. RASOR, Geometrical representation of functions of real and complex variables, 3 hrs.; Integral calculus, 5 hrs. By Assistant Professor HORTENSE RICKARD, Trigonometry, 5 hrs.; Analytic geometry, 5 hrs.

University of Wisconsin, June 28–August 6. By Professor A. DRESDEN, W. W. HART, E. P. LANE, H. W. MARCH, and E. B. SKINNER, and Messrs R. W. BABCOCK, J. E. DAVIS, and H. L. SMITH courses in Algebra; Analytic geometry; Differential calculus; Commerce algebra; Elementary solid geometry; Elementary mathematical analysis; Integral calculus; Teaching of secondary teachers; Differential equations; Theoretical mechanics; Modern analytic geometry; Differential geometry; Special topics in algebra; Elliptic integrals; Axioms of geometry; Point sets; and Differential equations of mathematical physics.

N.B. *University of Chicago:* The first term opens June *21* and not June *1* as stated *1920*, 192.

In view of the wide-spread and increasing interest in applied mathematics and the probability that a larger proportion of graduate students may in future be disposed to consider including technical, or semi-technical, subjects in their programs, the departments of Mathematics and Physics at the *Massachusetts Institute of Technology* announce the following list of courses available for graduate students. It will be understood that a particular course may be withdrawn in case a very small number of students should apply for it.

The departments will endeavor to carry out a periodic plan under which, while the more fundamental subjects may be given each year, others will be offered once in two or three years, in rotation. The possible range of subject matter may thus be capable of extension beyond the present list.

In addition to the titles given, there will also be opportunity—as heretofore—for the study of the usual courses in Advanced calculus, Theory of functions, Modern geometry, Modern analysis, etc., also experimental physics.

Analytical mechanics (MOORE). A problem course; three hours per week.
Mathematical laboratory (LIPKA). A treatment of alignment charts and other methods of graphical computation; two hours per week during the second and third terms.
Fourier's series (BAILEY). Two hours per week.
Application of mathematics to chemistry (HITCHCOCK). An application of thermodynamics to chemical problems; three times per week.
Mathematical theory of investment (TAYLOR). Three hours per week for one term.
Advanced mechanics (PHILLIPS). Two hours per week; including Analytical dynamics, Statistical mechanics and Theory of radiation.
Relativity and Einstein's theory of gravitation (PHILLIPS and MOORE). Two hours per week.
Heat conduction (PHILLIPS). Two hours per week for one term.
Aeronautics, 1st course (MOORE). Three hours per week.
Aeronautics, 2d course (WILSON). Two hours per week.
Electrochemistry (H. M. GOODWIN). Four hours per week.
Theoretical physics (WILSON). Three hours per week.
Electromagnetism (WILSON). Including (*a*) Electrodynamics, (*b*) Electromagnetic theory, (*c*) Applied electromagnetism, of which only one would be offered in a particular year.
Kinetic theory (W. S. FRANKLIN). Two hours per week for two terms.
Constitution of matter (WILSON). Two hours per week.
Research in mathematical physics (WILSON).

Each course will continue through the year (three terms) except where the contrary is stated.

MATHEMATICAL ASSOCIATION OF AMERICA.

The Council of the Association has adopted the following resolution which was proposed by the Council's Committee on a Mathematical Dictionary:

"Impressed with the possibilities for the more extensive development of pure and applied mathematics in America, and with the importance of such development to the nation, the Council records its conviction that there are undertakings whose active consideration would be highly desirable if adequate financial assistance might be regarded as available. Among such undertakings are: (1) the preparation and publication by societies or individuals of surveys, introductory monographs, translations, memoirs, and treatises in important fields, including the history of mathematics; (2) the organization of research fellowships; (3) the preparation and publication of an encyclopædia of mathematics in English; (4) the preparation and publication of an annual critical survey, in English, of the mathematical literature of the world; (5) the preparation and publication of a biographical and bibliographical dictionary of mathematicians."

This is supplementary to the resolution concerning a mathematical dictionary adopted by the Association at the Ann Arbor meeting. (See AMERICAN MATHEMATICAL MONTHLY, *1919*, 383–384.)

The Council has elected the following to membership in the Association, on applications duly certified:

LULA A. BARR, A.B. (Okla.). Dept. of math., Okla. College for Women, Chickasha, Okla.

H. S. BECKMAN, A.B. (Penna.). Computer, Ord. dept., U. S. Army, Washington, D. C.

WILLIAM BELLIS, B.S. (Chicago). Asso. prof., James Millikin Univ., Decatur, Ill.

W. E. CLELAND, A.M. (Pittsburgh). Instr., Univ. of Pittsburgh, Pittsburgh, Pa.

H. L. COAR, Ph.D. (Illinois). Prof. of math. and lecturer on astr., Marietta Coll., Marietta, Ohio.

J. S. COUNSELMAN, C.E. (Va. Polytech. Inst.). Head of dept. of math., Central High School, Birmingham, Ala.

F. W. DARLING, A.B. (Cornell). Geod. computer, Coast and Geod. Surv., Washington, D. C.

RUTH H. HALL, A.B. (Brown). Rosemary Hall, Greenwich, Conn.

H. D. HARPER. Instr., Murray Hill Voc. School, New York, N. Y.

MABEL E. HARRINGTON, A.M. (Brown). Computer, Ord. dept., U. S. Army, Kenilworth, D. C.

D. F. HEATH, A.B. (Illinois). Prof., Franklin Coll., Franklin, Ind.

R. S. HOAR, A.B., LL.B. (Harvard). Capt., C.A.; instr. in ballistics and ballistic math., Ord. School of Applic., Aberdeen Provg. Grd., Md.

ROSA L. JACKSON, A.B. (Western Coll.). Dean and head of dept. of math., Athens Coll., Athens, Ala.